ELEVENTH ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The eleventh annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado College, Colorado Springs, Colorado, on April 22, 23. There were forty present including the following twenty members of the association: G. H. Albright, A. G. Clark, E. A. Cummings, I. M. DeLong, Philip Fitch, A. J. Kempner, Claribel Kendall, G. H. Light, W. V. Lovitt, S. L. Macdonald, J. Q. McNatt, W. K. Nelson, Letitia Odell, O. H. Rechard, H. L. Rietz, W. J. Risley, H. E. Russell, Mary S. Sabin, C. H. Sisam, C. W. Wray.

The section voted to hold its next annual meeting at the Colorado School of Mines, Golden, Colorado. The following officers were elected: W. J. RISLEY, chairman; G. W. FINLEY, vice-chairman; PHILIP FITCH, secretary; G. H. LIGHT, treasurer.

At the complimentary dinner given by Colorado College on Friday, Dean C. B. Hershey delivered an address of welcome to which Professor W. J. Risley made the response. The section was favored Friday evening with an address "Group Insurance" by Professor H. L. Rietz of the University of Iowa. Professors H. E. Russell and C. H. Sisam read sketches of the lives and work of the late Dean H. A. Howe and the late Professor F. H. Loud respectively.

The following eleven papers were read:

1. "The sectioning of college freshmen in mathematics by means of the Iowa Placement Test" by Professor C. F. BARR.
2. "A grade weigher" by Professor G. H. ALBRIGHT.
3. "A graphic solution of tensions in cables" by Mr. J. Q. MCNATT.
4. "On a type of involutions in space" by Professor C. H. SISAM.
5. "Concerning the Heusler alloys" by Mr. PHILIP FITCH.
6. "An elementary method of solving matricial equations" by Professor O. H. RECHARD.
7. "Applications of elementary divisors" by Mr. D. L. GUNDER (by invitation).
8. "On a geometrical problem from the MONTHLY" by Professor A. J. KEMPNER.
9. "The Carus monograph on statistics" by Professor H. L. Rietz.
10. "Index number bias" by Professor W. V. LOVITT.
11. "A nomograph" by Professor W. K. NELSON.

In the absence of the author, the abstract of Professor Barr's paper was read by Professor Rechard. Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. The Iowa Training Examination in Mathematics was given at Purdue University to some 800 entering freshmen engineers, fall of 1925. This paper is a study of the degree of correlation between the grades obtained, and the subsequent classroom records in mathematics, of the persons involved. There was found a high degree of correlation. Furthermore, there appeared clearly defined lines for segregation into inferior, normal, and superior groups. The careful analysis of classroom records for the members of each group showed a remarkably high power of selectivity for the Iowa examination, and commends it as a tool for segregation of entering freshmen in mathematics. The conclusions are supported by the results of a similar study, made the preceding year, and ending simultaneously with the present study. While the paper demonstrates the possibility, it does not argue the advisability of classroom segregation of students.

2. In this paper Professor Albright described the construction and manipulation of a device for determining the average grade of a student involving the weighing of his various marks to correspond to the units of credit carried by his course. The device was of the type of a simple lever. Such an instrument was exhibited which would calculate the average with an error not exceeding three-hundredths of one percent.

3. Mr. McNatt gave a graphic solution of the tensions in a cable used to carry movable loads as in the case of aerial trams.

4. In this paper, the author discusses involutions in space such that the locus of the lines joining corresponding points is a congruence of order one defined by a (l, m) between the points on and the planes through a fixed line. Several particular involutions of this type were discussed at some length.

5. This paper dealt with mathematics as applied to data obtained from measurements on the resistance, permeability and thermo-electric affect of the Heusler alloys.

6. Professor Rechard showed that the solution of the general equation of the same order reduces to the problem of solving n^2 equations in n^2 unknowns, each equation being of degree n. A quadratic equation was used to illustrate the method.

7. Mr. Gunder showed that the principles of elementary divisors was directly applicable to the solution of problems of the dynamics of a particle where the particle is executing small vibrations about an equilibrium configuration. The study of this problem is greatly facilitated by the reduction of the equations of motion from the usual coordinates to normal coordinates.

8. Problem 3171(3167; 1926, 104) states that if, in an ellipse, a straight line is drawn through one Focus F_1, its intersection P_1 with the circumference joined with the other Focus F_2, the intersection P_2 of P_1F_2 with the ellipse again joined with F_1, etc. etc., the limiting position of the straight line will be

the straight line joining F_1F_2. It is shown that this property is in no way characteristic of the ellipse. An analogous property exists, for example, for all closed convex curves.

9. Professor Rietz discussed first the question of making statistical theory more available to the general mathematical reader by means of the third Carus Mathematical Monograph. He then divided the problems considered in the monograph into two general classes. In problems of the first class, the main interest centers around the properties of a random sample drawn from a "population." In problems of the second class, the main interest centers around the question of making and checking the validity of statistical inferences about the population from which the sample is drawn. In dealing with the sample we introduce very early the concepts of relative frequency, arithmetic mean, various other averages, and certain measures of dispersion. In considering the population, we introduce the parallel concepts of probability, mathematical expectation of the value of a variable, and of the powers of the deviations of a variable from its expected value. After discussing these concepts, the paper gives a summary of the material of the monograph dealing with the following three topics of dominant interest in recent progress in statistics: Generalization of frequency curves, correlation theory, and random sampling theory.

10. Definite proofs were made as to the presence or absence of bias for the weighted arithmetic average of relatives for the weights I p_0q_1, II p_0q_1, III p_iq_i, III p_iq_0, IV p_iq_i. Definite proofs were given as to the relative size of the index numbers with weights I, II, III, IV.

11. In this paper a description was given of a nomograph of an approximation to the formula for the relation between effective and nominal interest rates.

For given values of j and p, i could be read accurately to at least four significant figures.

PHILIP FITCH, *Secretary*

THE TEACHING OF MATHEMATICS IN GERMAN SECONDARY SCHOOLS AND THE TRAINING OF TEACHERS FOR THESE SCHOOLS

By CHARLES A. NOBLE, University of California, Berkeley

The statements which follow have to do primarily with conditions in Prussia. In the other German states the departures from the Prussian system are, in general, inconsiderable. The head of the Prussian school system is the Minister of Science, Art, and Popular Culture (Minister für Wissenschaft, Kunst, und Volksbildung). The regulations governing all educational institutions are issued by him; but the immediate control of the schools of any province is entrusted to the Provincial Schoolboard (Provinzial Schulkollegium) of that province.

There are now, as before the war, three main types of secondary school in Germany. These are the Gymnasium, or classical school; the Realgymnasium, or Latin-modern-language school; and the Oberrealschule, or mathematics-natural science school. They are all nine-year schools. The revolution abolished the three-year preparatory school (Vorschule) which was attached formerly to each of the three types, and set up the universal four-year preparatory school (Grundschule), upon which every higher German school rests. Every German child enters the Grundschule at six years of age. The normal ages in the full secondary school are thus from ten to nineteen. Graduation from the secondary school automatically admits the pupil to the university. At that stage the pupil's training is at least two years beyond that of the graduate of an American high school, as to breadth, and still further ahead as to thoroughness.

The official guide for the Prussian secondary school is the ministerial edict, issued in April, 1925, "Richtlinien für die Lehrpläne der höheren Schulen Preussens." According to this edict the aims of the secondary schools appear to be, briefly, as follows: The particular problem of the Gymnasium is, through the study of Greek and Latin, its characterizing subjects, to prepare the pupil for the appreciation of history and the mother tongue and to discipline his mind by a method of work which he can apply in other fields; that of the Realgymnasium is, through the study of English and French, reinforced by Latin, to prepare the pupil to understand German culture as related to Western culture and as influenced by mathematical and scientific thought; that of the Oberrealschule is, through the greater prominence given to mathematics and

natural science, to unfold to the pupil the possibilities that lie in these subjects for humane culture.

The secondary schools for girls have the same aims and problems as those for boys. The six-year Lyzeum is the foundation of all types. Corresponding, respectively, to the Gymnasium, the Realgymnasium, and the Oberrealschule are the gymnasiale Studienanstalt, the Oberlyzeum, and the Oberlyzeum der Oberrealschulrichtung.

The curriculum of each type of school is heavy and is substantially prescribed. It is, however, less inflexible than formerly, because reform has introduced subsidiary types designed to permit a greater variety of choice.

It may be interesting to set out the curriculum in mathematics for, say, the Prussian Gymnasium, the secondary school in which that subject is least heavy. The "Richtlinien" prescribe as follows:

VI. (Age 10)[1]

The four fundamental rules with integers, German weights, measures, and coins. Practice in writing and calculating with decimals. Applications to home and community life. Intuitional treatment of space-forms with especial reference to the needs of instruction in physical geography.

V. (Age 11)

Continued practice with decimals. Divisibility of numbers. Four fundamental rules with fractions. Representation of numbers by means of line segments and areas. Rule of three. Applications to home, community, and state. Continued study of space-forms.

IV. (Age 12)

Arithmetic. Four fundamental rules with decimals; changing common fractions into decimals and conversely. Short methods. Simplest cases of percentage, discount, and interest, and other problems of everyday life, with application of the rule of three. Representation of number sequences by means of line segments and areas. Use of tables, especially calculation of mean values and ratios. Applications to the life of the community and the state.

Geometry. Intuitional development of fundamental notions; sides and angles of triangles. Simplest triangle constructions. Congruence theorems.

Geometrical Drawing and Measurement. Practice in the use of ruler, drawing triangles, and compasses. Drawing of parallels and perpendiculars and of nets of cubes, parallelopipeds, tetrahedrons, and octahedrons. Orthogonal projection of cube and parallelopiped. Construction of the space-diagonals of these

[1] The normal age for admission to "sexta" is ten; it is possible and not unusual, however, for a pupil to finish the Grundschule in three years in which case he enters "sexta" at nine years of age, as formerly.

bodies. Construction of prisms and pyramids and of nets of these solids. Measurement of segments and angles.

UIII (Age 13).

Algebra. Introduction of operations with letters. Four fundamental operations with integral and with fractional numbers. Representation by means of segments and rays. Calculation of tables from formulas. Simple equations of first degree with one unknown, in connection with operations with rational numbers. Introduction of linear function, if possible. Drawing of curves.

Geometry. Completion of the study of triangles. Continued construction of triangles. Theory of quadrilaterals, especially parallelograms and trapezoids. Calculation and comparison of areas (Pythagorean theorem). Extension of geometrical considerations and measurements to space.

Geometrical Drawing and Measurement. Projection of points, line segments, and plane figures upon the plane. Angle of inclination of line and of plane. Intersections of two planes whose traces and inclinations are given. Pyramids. Roofs. Measure of segments and angles.

OIII (Age 14).

Algebra. Equations of first degree with one or more unknowns; simple applications, especially from everyday life. Introduction of graphical representation of empirical functions; representation of linear functions (the linear function as straight line) and their use in solving equations of first degree. The function $y=mx$. Use of millimeter paper.

Geometry. Circles. Further study of space-forms.

Geometrical Drawing and Measurement. Simple masses of shrubbery and borders of garden paths. Measurement of segments, angles, and areas.

UII (Age 15).

Algebra. Powers with positive and negative integral exponents. The function $y=x^n$, n positive or negative, and its graphical representation, especially the graphs of the general parabola and of the rectangular hyperbola. Operations with radicals, and their graphs. Extraction of square root. Exponential and logarithmic functions. Inverse of a function. Four place logarithmic table and the use of logarithms. The slide rule. Quadratic equations with one unknown.

Geometry. Equality of ratios between segments; theory of similarity. Application to circle and right triangle. Length and area of circle. Selections from the history of geometrical problems (e.g. quadrature of the circle). Calculation of simplest figures.

Geometrical Drawing and Measurement. Tri-rectangular axes; the regular bodies in axonometric representation. Curve drawing in connection with algebra. Approximate constructions (circle). More exact measurements (nonius). Surveying in connection with the theory of similarity.

OII (Age 16).

Algebra. Simple integral and rational functions. Simple equations, and systems of equations which can be solved by quadratic equations—numerical and graphical treatment. Arithmetic and geometric series. Infinite geometric series. Compound interest and bonds, with applications from commercial life (political arithmetic). Binomial theorem with positive integral exponent.

Geometry. The trigonometric functions. Simple triangle calculations. Goniometry. Notion of periodic function. Geometrical calculations continued.

Geometrical Drawing and Measurement. Projection of the circle. Constructions for trigonometric problems, including those whose data are not all in one plane. Curve drawing in connection with algebra. Simple exercises in surveying and leveling.

UI (Age 17) and OI (Age 18).

Algebra. Introduction to infinitesimal calculus. Definition of differential quotient, its geometrical and its physical significance; its application to rational and, if possible, to trigonometric functions, especially in the calculation of greatest and least values, points of inflexion, inflexional tangents, etc. Simplest exercises in finding areas and volumes with the aid of integral calculus (e.g. sphere, paraboloid, etc.). Construction of the number system, from the positive integer to the complex number. Simple representations by means of functions of a complex variable.

Geometry. Review of the curvilinear figures thus far studied and the introduction of analytic geometry as far as the general equation of second degree. Supplementary theorems from stereometry (sphere). Straight line and plane in space. Plane and cone. Fundamental notions of spherical trigonometry (sine theorem and cosine theorem). Applications to geodesy and astronomy.

Geometrical Drawing and Measurement. Fundamental problems on the point, line, and plane. Conic sections. Projection of the sphere. Simple astronomical observations with measurements and calculations.

Review from historical and philosophical points of view.

The following tabular program of the Prussian Gymnasium indicates the number of hours devoted each week to each subject during the nine years. It is taken from the Ministerialerlass of October 31, 1924, Berlin. The "hours" are about forty-five minutes in length.

	Gymnasium									
	VI	V	IV	UIII	OIII	UII	OII	UI	OI	Total
Religion	2	2	2	2	2	2	2	2	2	18
German	5	4	3	3	3	3	4	3	3	31
Latin	7	7	7	6	6	5	5	5	5	53
Greek	–	–	–	6	6	6	6	6	6	36
Modern Language	–	–	3	2	2	2	2	2	2	15
History	–	1	2	2	2	3	3	3	3	19
Physiography	2	2	2	1	1	1	1	1	1	12
Mathematics	4	4	4	3	3	4	3	4	4	33
Natural Science	2	2	2	2	2	2	2	2	2	18
Drawing	2	2	2	2	2	1	1	1	1	14
Singing	2	2								4
Total	26	26	27	29	29	29	29	29	29	253

In addition to the above there is a weekly prescription of: 4 hours for physical exercise during the entire course of nine years; 4 hours for music during the six years IV to OI inclusive; 6 hours for Arbeitsgemeinschaften during the three years OII to OI inclusive.

It will be seen that the weekly program of the German pupil contains at least fifty per cent more hours than that of the American high school pupil.

Another striking difference is that, whereas in American schools plane geometry is studied daily for one year, to the exclusion of all other mathematics, and similarly for algebra, in Germany all the types of mathematics are studied during the same year. Algebra including arithmetic, and geometry including trigonometry are carried on abreast or in close alternation, at the discretion of the teacher, according as they supplement one another.

The "Richtlinien" of 1925 not only define the aims and prescribe the programs of the secondary schools, but they set out a very considerable body of principles and of suggestions as to method. They insure the same goal and standard for all schools of like type and facilitate the movement of pupils from one school to another. They aim to preserve unity in the variety of secondary schools by placing certain fundamental subjects, viz. religion, German, history, and physiography in the center of all secondary school programs and they demand that each school shall accomplish its special task through the interaction of its characteristic subjects with these fundamental subjects. They insist that the training should move toward an organic whole rather than be merely teaching of separate subjects, as formerly; that teaching should be teaching to work, and not the imparting of information; that the teacher should continually watch for those powers of the pupil which can be developed under his guidance; that the teacher's problem is to make the bridge between the acquirement of certain knowledge, without which high mental activity is impossible, and the acquirement of the power of independent work, without which knowledge is sterile.

During three and a half weeks of the summer of 1926, I visited mathematics classes in Göttingen, Berlin, Dresden, Stuttgart, and Hamburg. This brief time sufficed to give certain impressions as to personality of a considerable number of teachers, their methods of teaching, and its effectiveness. These teachers, always well trained, were nearly always men and women of good presence and pleasing personality who had the confidence of their pupils, with whom the relation was rooted in the teacher's desire to be helpful.

I saw no provision in any class for the separation, into different sections, of pupils of differing ability, as is done in English schools. While the better pupils are stimulated by the offer of more difficult tasks, the Arbeitsgemeinschaft provides in the later years the fuller opportunity for gifted pupils. In it pupils who show especial interest in some subject of the curriculum may, under the informal guidance of a teacher, widen their knowledge and test their ability.

Two factors which contribute to the excellence of the training in German schools are that mathematics is taught in close touch with physics, frequently by the same teacher, and, secondly, that geometrical drawing is taught throughout the entire nine years. This assists materially in the cultivation of the space sense. It is noteworthy also that geometry is begun in the first year of the school, when the pupil is ten years old, earlier than is usually thought possible in America. This instruction is, at first, only intuitional—demonstration comes later—and the pupil is at once encouraged to enlarge his notions of three dimensional space, the space in which he has had experience. One excellent teacher declared to me that it was absurd to start a boy with plane figures when solids are what the child first sees. He sometimes found, with his maturer pupils, that their perception with solid figures was immediate in problems where he was puzzled, because he had been trained in the old way which suppresses and warps the child's experience.

In a Hamburg school I found projective geometry being taught to two classes in UI, corresponding roughly, to our freshman year. I was told that this was the only school in Germany where this subject was taught. The University of California is one of the few institutions in America, if not the only one, where projective geometry is offered to freshmen. In both cases the situation arose from the presence of a stimulating teacher who was an enthusiast for that subject.

The teaching in all the schools which I visited was marked by more emphasis upon careful, detailed attention to theoretical development and less upon illustrative numerical problems than is the case in American schools and colleges. This was as noticeable in the Oberrealschule, where future engineers would be enrolled, as in the classical schools.

The recent school reform in Germany arose, according to some, in a desire to overcome the materialism of the German people, which was thought to be due to too much concern with intellectual things, too little with things of feeling. A corrective was sought in a plan which should direct less attention to science and mathematics, more to music and art; which should give prominence to modern language in order that the Germans, through reading foreign literature, could come into touch with other peoples; and which should improve the tone of the youth by fostering an interest in open air recreation, in "Spiel und Sport."

Let us now consider the training of teachers of mathematics for the secondary schools of Prussia.

1. The prospective teacher must have graduated from a nine-year secondary school in Prussia, or have had equivalent training. This is in general fully, in some respects more than the equivalent of the first two years at an American university.

2. The prospective teacher must have passed the state examination (wissenschaftliche Prüfung) set for candidates for a teaching position. This examination must be in two major subjects, one minor subject, and in philosophy. The examination is held by a committee composed of university professors and schoolmen. To qualify for admission to this examination the candidate must have studied at least eight semesters at a university, of which at least six must have been at a German, and at least three at a Prussian university. He must submit evidence that the studies in the majors have included the appropriate laboratory and seminary courses and that during two semesters he had training in physical exercise, both theoretical and practical.

The examination consists of a written and an oral part. In the former the candidate submits two papers upon assigned subjects, one from the field of the chief major, the other from that of the other major or from philosophy, at his option. He is allowed five months to prepare these papers. The candidate is examined orally in each of his subjects.

3. The prospective teacher, after having passed the state examination, must have spent two years in practice preparation, for which he is assigned to a secondary school recognized for that purpose, and during which he is under the guidance of experienced secondary school teachers.

4. The prospective teacher must pass the pedagogy examination (Pädagogische Prüfung), to qualify for which he must submit a favorable report upon his work during the two practice years. This examination embraces a written part consisting of a thesis on a topic from the theory of education, which shall exhibit his own views acquired during his two practice years and be supported by scientific argument; two test class lessons given in the

presence of the examining committee; and an oral part designed to test his grasp of actual teaching problems.

There is no provision at the universities and technical schools for future teachers to acquire, during their scientific training, detailed drill in the experimental work of elementary science and descriptive geometry which is necessary in secondary school teaching. To supply this lack, summer courses are offered at universities and other institutions for the benefit of those actually engaged in teaching. Such courses, for Prussia, are under the auspices of the Staatliche Hauptstelle für Naturwissenschaftlichen Unterricht, in Berlin. In the courses which I visited in chemistry, physics, and descriptive geometry, secondary school teachers were actually performing, under the direction of experienced university and school men, experiments appropriate for secondary schools, much as is done at our university summer courses for teachers. The apparatus was crude and the accommodations in other respects were meager, due to that widespread poverty which, since the war, has been a blighting handicap in German educational undertakings. University and school men cannot afford to buy books or even the scientific magazines so necessary to keep them abreast of progress; libraries cannot make their needful purchases; many school children are unable to buy the prescribed books; laboratories are short of supplies. In spite of such disabilities, perhaps because of them, there is a close-knitting bond uniting all the school agencies and a spirit of cooperation between community and schools which enables them to carry on and to do admirable work.

THE "MYSTIC" NUMERAL 7

By HARRIS HANCOCK, University of Cincinnati

In 1892 a writer in one of the Berlin newspapers had learned that the integers in the number $N_1 = 142857$ repeat themselves in cyclic order when N_1 is multiplied respectively by 1, 3, 2, 6, 4, 5.[1] For example, $142857 \times 3 = 428571$. Noting the fact that $142857 \times 7 = 999999$ and that three 9's result from adding the digits in pairs $(1+8=9;\ 4+5=9;\ 2+7=9)$, this writer, possibly a fundamentalist, connected the above results with the Holy Trinity. Dividing 1 by 49, I wrote down the 42 numerals that occur before the decimal begins to repeat itself (as given below), indicated that this number repeated itself cyclically when multiplied by the 42 integers that are less than and relatively prime to 49 and queried what was its biblical analogon.

Observe first that, when divided by 7, the remainders of 10, 30, 20, 60, 40, 50, are 3, 2, 6, 4, 5, 1, so that $1/7 = .142857$. It may be remarked parenthetically

[1] Professor Robert E. Moritz, in a paper, *On products whose digits are cyclical permutations of the digits of the multiplicand*, has given recently an interesting theorem from which this peculiar property of the number 142857 follows as a very special case. This Monthly, vol. 34 (1927), pp. 33–36.

that these remainders repeat themselves cyclically (mod 7) when multiplied by 1, 3, 2, 6, 4, 5. Due to the fact that a circulating decimal is a cyclic expression, it follows that 1/7, 3/7, 2/7, 6/7, 4/7, 5/7 are equal to circulating decimals with the same digits occurring cyclically. And these circulating decimals are nothing other than the results of multiplying N_1 by 1, 3, 2, 6, 4, 5. Observe that, otherwise interpreted, this simply means that $10^6 \equiv 1 \pmod 7$ and that corresponding to a definite power of 10 there is a definite remainder and vice versa. Observe next that $10^{42} \equiv 1 \pmod{49}$ and that 42 is the lowest power after zero that satisfies this congruence. For, writing $10^k \equiv 1 \pmod{49}$ it follows that $10^k \equiv 1 \pmod 7$ so that k is a multiple of 6, say $k = 6h$. Hence $(10^{6h}) \equiv 1 \pmod{49}$; or, since $10^6 - 1 = 7t$, where $t \not\equiv 0 \pmod 7$, it is seen that

$$(1 + 7t)^h \equiv 1 \pmod{49} \quad \text{or} \quad 1 + 7ht + \frac{h(h-1)}{1 \cdot 2}(7t)^2 + \cdots \equiv 1 \pmod{49}.$$

Accordingly we must have $h \equiv 0 \pmod 7$. Similarly, $s = 6 \cdot 7^2$ is the lowest power of 10 which satisfies $10^s \equiv 1 \pmod{7^3}$ and in general $s = 6 \cdot 7^{r-1}$ is the lowest power of 10 which satisfies $10^s \equiv 1 \pmod{7^r}$, where r is a positive integer. And corresponding to a definite power there is a unique residue.

To determine the forty-two digits which compose a period of the circulating decimal for $1 \div 7^2$, we may divide by 7 the circulating decimal D given in the following array. The number N_2 is the quotient.

	(I)	(II)	(III)	(IV)	(V)	(VI)	(VII)
$D = .142857$	142857	142857	142857	142857	142857	142857	142857
	1	2	3	4	5	6	0
$N_2 = .020408$	163265	306122	448979	591836	734693	877551	

The following table (to be described later) is inserted here for convenience.

	3	0	5	4	4	5	1
	4	1	6	5	5	6	2
N_3	5	2	0	6	6	0	3
	6	3	1	0	0	1	4
	0	4	2	1	1	2	5
	1	5	3	2	2	3	6
	2	6	4	3	3	4	0

Returning to the number N_2, notice that in the first interval (I) the final remainder is 1 which is written above the number; the final remainder in the second interval (II) is 2; in the third interval (III) it is 3; $\cdots$; and in the seventh interval (VII) this remainder is 0, showing that the decimal expressing $1 \div 7^2$ begins then to repeat. Next observe that, corresponding to every digit in N_2, there is a second digit such that the sum of the two digits is 9, so that the sum of the digits of N_2 is $21 \cdot 9 = 7 \cdot 3^3$. Similarly in N_r, the sum of the digits is $7^{r-1} \cdot 3^3$.

It is of interest to observe the sequence of the remainders 1, 2, 3, 4, 5, 6, 0, just noticed. To do this note that

$$10 \equiv 3\ ;\ 10^2 \equiv 2\ ;\ 10^3 \equiv 6\ ;\ 10^4 \equiv 4\ ;\ 10^5 \equiv 5\ ;\ 10^6 \equiv 1\ ;\ (\text{modulo } 7).$$

Hence, if $N = \cdots n_6 n_5 n_4 n_3 n_2 n_1 n_0$, where n_0 is the numeral in the units' place, n_1 the numeral in the ten's place, etc., it is seen that

$$\begin{aligned} N &= n_0 + 10n_1 + 10^2 n_2 + 10^3 n_3 + 10^4 n_4 + 10^5 n_5 + 10^6 n_6 + \cdots \\ &\equiv n_0 + 3n_1 + 2n_2 + 6n_3 + 4n_4 + 5n_5 + n_6 + \cdots \\ &\equiv n_0 + 3n_1 + 2n_2 - n_3 - 3n_4 - 2n_5 + n_6 + \cdots \pmod 7 \end{aligned}$$

or

$$\begin{aligned} N \equiv (n_0 - n_3 + n_6 - n_9 + \cdots) + 3(n_1 - n_4 + n_7 - n_{10} + \cdots) \\ + 2(n_2 - n_5 + n_8 - n_{11} + \cdots) \pmod 7. \end{aligned}$$

It is thus seen that

$$142857 \equiv 7 - 2 + n_6 + 3(5 - 4) + 2(8 - 1) \equiv 1 + n_6 \pmod 7.$$

Hence, in the first interval in the array above, where $n_6 = 0$, the remainder is 1; and as $n_6 = 1$ for the second interval, the remainder in that interval is 2, etc.

To determine the number N_3 which corresponds to $1 \div 7^3$, write down the number N_2 consisting of forty-two digits (corresponding to $1 \div 7^2$) seven times, each of these seven intervals being written the one over the other as in the diagram above. Observe that, if N_2 is divided by 7, the remainder after the sixth digit is 3; the remainder after the twelfth digit is 0, etc.; the remainder after the 42nd digit is 1. Continuing with this remainder into the line below, the next remainder is 4, etc. . . . . ; the final remainder on the seventh line being 0, showing that the period is ended. It is again of interest to note the sequence of remainders which follow the one below the other in the above scheme. Of course, when N_3 is multiplied by any multiple of 7 that is less than 7^3, we have seven periods of digits in cyclic order that compose N_2; and if again these are multiplied by a multiple of 7 that is less than 49, we have 49 periods of the digits that cyclically constitute N_1. Further, if N_3 be multiplied by 7^3

we have $6 \cdot 7^2$ consecutive 9's; and if N_2 is multiplied by 7^2, there are 42 consecutive 9's.

Bachman, on p. 381 of vol. 36 (1891) of the Zeitschrift für Mathematik und Physik writes that Sturm had communicated to him the cyclic properties of N_1 when multiplied by 3, 2, 6, 4, 5, 1, and that these results were taken from a French periodical (Zeitschrift). J. C. Burkhardt, in his *Tables des diviseurs pour tous les nombres du premier million* (Paris, 1817), p. 114, states that 10 is a primitive root of 148 of the 365 primes p, where $5 < p < 2500$. For example $10^{16} \equiv 1 \pmod{17}$, and $1/17 = .0588235294117647$. Here again among the digits that constitute this decimal, for every 0 there is a 9, for every 5 there is a 4, etc. so that the sum of these digits is $8 \cdot 9$; the sum of the digits in the decimal that expresses $1/17^2$ is $8 \cdot 17 \cdot 9$ while the sum of those in $1/17^3 = 8 \cdot 17^2 \cdot 9$, etc. L. E. Dickson, in his *History of the Theory of Numbers*, vol. 1, p. 161, gives a reference to Goodwyne who, Dickson says, noted that when $1/p$ is converted into two half-periods, the sum of corresponding quotients is 9.

INTERPOLATION FORMULAS DEPENDENT UPON THE UNDERLYING FUNCTION

By JOHN F. REILLY, University of Iowa

We are accustomed to the use of interpolation formulas for finding the approximate value of a function $f(x)$ at $x = x'$, even when we can calculate $f(x')$ directly to any desired degree of accuracy. The approximate value found by interpolation is sufficiently accurate for many purposes, and the labor involved in the direct calculation is thereby eliminated.

The Gregory-Newton interpolation formula

$$y_x = y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \cdots$$

proceeds without invoking very directly the properties of the underlying function; it merely makes use of a limited number of values of the function. Thus we would interpolate in the same way when the values (1, 2), (2, 4), and (3, 8) have underlying them the exponential function $y = 2^x$, as when they have underlying them the algebraic function $y = x^2 - x + 2$. The interpolation would accurately reproduce the latter function but not the former. It is the purpose of this paper to develop a scheme of interpolation that is dependent upon the underlying function. This is accomplished by developing formulas containing one or more parameters, and then using the underlying function to determine these parameters.

Denote the values of the function y corresponding to the values 0, 1, 2, 3, of the argument x, by y_0, y_1, y_2, y_3 respectively. It is clear that the slopes of the straight lines P_0P_1, P_1P_2, and P_2P_3, are y_1-y_0, y_2-y_1, and y_3-y_2 respectively. To obtain an interpolation formula for use between y_1 and y_2, we will impose the following conditions: that the formula reduce to y_1 when $x=1$, and to y_2 when $x=2$; that the slope of its graph at P_1 be $k(y_1-y_0)+l(y_2-y_1)$, where $k+l=1$, and that the slope at P_2 be $m(y_2-y_1)+n(y_3-y_2)$, where $m+n=1$. These slopes are arbitrarily weighted arithmetic means of the slopes of the straight lines P_0P_1, P_1P_2, and P_2P_3.

Assuming the interpolation formula to be of the form

$$y = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 \cdots$$

we find the slope to be $y' = a_1 + 2a_2(x-1) + 3a_3(x-1)^2$.

Imposing the above conditions leads to the following values of the coefficients in terms of y_0 and its leading differences:

$$a_0 = y_0 + \Delta y_0, \qquad a_2 = (2 - n - 2l)\Delta^2 y_0 - n\Delta^3 y_0,$$

$$a_1 = \Delta y_0 + l\Delta^2 y_0, \quad a_3 = (-1 + n + l)\Delta^2 y_0 + n\Delta^3 y_0.$$

Substituting these values of the coefficients, we obtain after reduction

$$(A) \quad y = y_0 + x\Delta y_0 + (x-1)[l(x-2)^2 + n(x-1)(x-2) - (x-1)(x-3)]\Delta^2 y_0 + n(x-1)^2(x-2)\Delta^3 y_0,$$

a formula containing two parameters l and n. We note the three following special cases:

When $l=n$, formula (A) becomes

$$(B) \quad y = y_0 + x\Delta y_0 + [n(x-1)(x-2)(2x-3) - (x-1)^2(x-3)]\Delta^2 y_0 + n(x-1)^2(x-2)\Delta^3 y_0.$$

When $n=\frac{1}{2}$, formula (B) becomes

$$(C) \quad y = y_0 + x\Delta y_0 + \frac{x(x-1)}{2}\Delta^2 y_0 + \frac{(x-1)^2(x-2)}{2}\Delta^3 y_0.$$

This is Sprague's third differences osculatory formula.[1]

[1] Journal of the Institute of Actuaries, vol. 41 (1907), p. 545. This is what we should expect, since in this case Sprague's partial interpolation curves are parabolas, and when $l=n=\frac{1}{2}$ the assumed slopes at P_1 and P_2 are those of the lines joining P_0P_2 and P_1P_3 respectively. It is well known that in a parabola a chord is parallel to the tangent at the point midway between the ends of the chord.

When $l=\frac{2}{3}$, $n=\frac{1}{3}$, formula (A) becomes[1]

$$(D) \qquad y = y_0 + x\Delta y_0 + \frac{x^2-1}{3}\Delta^2 y_0 + \frac{(x-1)^2(x-2)}{3}\Delta^3 y_0.$$

In the application of formula (A) or formula (B), the parameters will be determined so that the interpolated values will reproduce as nearly as possible the underlying function. To illustrate the use of formula (B), consider the function $(1+x)^r$ where four values corresponding to equidistant values of x are known, and where its value is required for some x in the middle interval. The value of the parameter n in formula (B) that will produce the best interpolated value will depend upon r. In practice one would compute the value of n corresponding to the value of r in which he was interested. If by the best interpolated value we mean the best in the least squares sense, we determine n so that the sum of the squares of the differences between the interpolated values as given by formula (B) and the actual values as given by $(1+x)^r$, for all values of x over the range x_1 to x_2, is a minimum. This requires the determination of n so that the definite integral

$$\int_{x_1}^{x_2} \{(1+x)^r - y_0 - x\Delta y_0 - [n(x-1)(x-2)(2x-3) - (x-1)^2(x-3)]\Delta^2 y_0 - n(x-1)^2(x-2)\Delta^3 y_0\}^2 dx.$$

shall have a minimum value.

If we write $f_1(x)$ for $(1+x)^r - y_0 - x\Delta y_0 + (x-1)^2(x-3)\ \Delta^2 y_0$, and $f_2(x)$ for $-(x-1)(x-2)(2x-3)\Delta^2 y_0 - (x-1)^2(x-2)\Delta^3 y_0$, this definite integral may be written

$$\int_{x_1}^{x_2} [f_1(x) + nf_2(x)]^2 dx.$$

Sufficient conditions for a minimum are

$$\int_{x_1}^{x_2} [f_1(x) + nf_2(x)]f_2(x)dx = 0, \text{ and } \int_{x_1}^{x_2} [f_2(x)]^2 dx > 0.$$

The inequality is satisfied except in the case where $f_2(x)$ is identically zero over the range x_1 to x_2, in which case formula (B) is linear in x. When $f_2(x) \not\equiv 0$ we find

$$n = \int_{x_1}^{x_2} f_1(x)f_2(x)dx \div \int_{x_1}^{x_2} [f_2(x)]^2 dx.$$

This value of n substituted in formula (B) will give the best reproduction of $(1+x)^r$ over the range x_1 to x_2, under the least squares criterion adopted.

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[1] Reilly, Proceedings of the Iowa Academy of Science, vol. 31 (1924), p. 369.

When applying formula (A), which contains two parameters l and n, the method of least squares requires that the definite integral

$$\int_{x_1}^{x_2} \{(1+x)^r - y_0 - x\Delta y_0 - (x-1)[l(x-2)^2 + n(x-1)(x-2) - (x-1)(x-3)]\Delta^2 y_0 - n(x-1)^2(x-2)\Delta^3 y_0\}^2 dx$$

have a minimum value.

If we write $f_1(x)$ for $(1+x)^r - y_0 - x\Delta y_0 + (x-1)^2(x-3)\Delta^2 y_0$, $f_2(x)$ for $-(x-1)(x-2)^2\Delta^2 y_0$, and $f_3(x)$ for $-(x-1)^2(x-2)(\Delta^2 y_0 + \Delta^3 y_0)$, this definite integral may be written

$$\int_{x_1}^{x_2} [f_1(x) + lf_2(x) + nf_3(x)]^2 dx.$$

Sufficient conditions for a minimum are

(1) $\int_{x_1}^{x_2} [f_1(x) + lf_2(x) + nf_3(x)]f_2(x)dx = 0$, (3) $\int_{x_1}^{x_2} [f_2(x)]^2 dx > 0$,

(2) $\int_{x_1}^{x_2} [f_1(x) + lf_2(x) + nf_3(x)]f_3(x)dx = 0$, (4) $\int_{x_1}^{x_2} [f_3(x)]^2 dx > 0$,

(5) $$\left[\int_{x_1}^{x_2} f_2(x)f_3(x)dx\right]^2 < \int_{x_1}^{x_2} [f_2(x)]^2 dx \times \int_{x_1}^{x_2} [f_3(x)]^2 dx.$$

The inequalities (3) and (4) are satisfied except, as before, when $f_2(x)$ and $f_3(x)$ are identically zero over the range x_1 to x_2, and the equations (1) and (2) will determine values of l and n which will make the definite integral a minimum, provided the inequality (5) is also satisfied.

Such values of l and n substituted in formula (A) will give in this case the best reproduction of the function $(1+x)^r$ over the range x_1 to x_2, in the sense of the least squares criterion.

In the development of the above formula we have employed four values of the function, and hence third order differences, but it is obvious that other formulas could be similarly developed containing either a lesser or a greater number of differences. When this direct appeal is made to the underlying function it seems reasonable to believe that there would be a gain in accuracy in the interpolated values with the use of a given number of differences, or that a given degree of accuracy is attainable with a lesser number of differences.

NOTE ON THE UPPER LIMIT TO THE VALUE OF A DETERMINANT

By HARVEY A. SIMMONS, Northwestern University.

The purpose of this note is to obtain by an elementary method a new expression for the upper limit to the value of a determinant whose elements are real numbers—an expression which never exceeds the Hadamard upper limit,[1] $n^{n/2}M^n$, where n is the order of the determinant and M is a fixed, positive constant such that $|a_{ij}| \leqq M$ for every element a_{ij} of the determinant.

Example. If $n=4$ and $M=1$, Hadamard's upper limit is $4^2=16$, which can actually be attained.[2]

We use the notation

$$(1) \qquad D \equiv \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ \cdots\cdots\cdots \\ a_{n1} & a_{n1} \cdots a_{nn} \end{vmatrix}, \quad \Delta \equiv \begin{vmatrix} A_{11} & A_{12} \cdots A_{1n} \\ \cdots\cdots\cdots \\ A_{n1} & A_{n2} \cdots A_{nn} \end{vmatrix},$$

where each a_{ij} $(i, j=1, 2, \cdots, n)$ is a real number and A_{ij} is its cofactor in D, so that Δ is the adjoint of D, and where

$$(2) \qquad |a_{ij}| \leqq M_i, \qquad \sum_{j=1}^{n} a^2_{ij} = n_i\lambda_i^2,$$

for every i; M_i being the maximum absolute value of any element in the ith row of D, λ_i being a fixed, positive constant, obviously less than or equal to M_i, and n_i denoting the number of non-zero elements in the ith row of D.

Let the elements in the ith row of D vary in such a way that the value of D remains unchanged. On differentiating D with respect to the elements in its ith row, we obtain

$$(3) \qquad \sum_{j=1}^{n} \frac{\partial D}{\partial a_{ij}} \cdot \delta a_{ij} = 0 \quad \text{for every } i,$$

where the δa_{ij} are variations which are subject only to the condition

$$(4) \qquad \sum_{j=1}^{n} a_{ij}\,\delta a_{ij} = 0, \text{ which is the result of differentiating}$$

the second of equations (2). Now $\partial D/\partial a_{ij} = A_{ij}$ and we have, by (3) and (4),

$$(5) \qquad A_{ij} = \partial D/\partial a_{ij} = ka_{ij}, \text{ where } k \text{ is a constant.}$$

From the fact that $\sum a_{ij}A_{ij}=0$ for each i when j runs from 1 to n, and equation (5), we conclude that

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[1] Bulletin des Sciences Mathématiques, (2), vol. 17, p. 240.

[2] Loc. cit., p. 244. The author proves that if n is a power of 2 or if n is 12 or 20, the upper limit can be attained by determinants whose elements are real numbers.

$$k \sum_{j=1}^{n} a_{ij}^2 = D \text{ or } k = D/(n_i \lambda_i^2). \tag{6}$$

Now substituting in (5) the value of k given by (6), we have $A_{ij} = a_{ij} D/n_i \lambda_i^2$. Substituting this value of A_{ij} in (1) and making use of the well-known theorem that $\Delta = D^{n-1}$, we obtain $\Delta = D^{n+1} \div n_1 n_2 \cdots n_n \lambda_1^2 \lambda_2^2 \cdots \lambda_n^2 = D^{n-1}$. From this equation, either $D = 0$ or $D^2 = n_1 n_2 \cdots n_n \lambda_1^2 \lambda_2^2 \cdots \lambda_n^2$. Hence the absolute value of D never exceeds $E \equiv (n_1 n_2 \cdots n_n)^{1/2} \lambda_1 \lambda_2 \cdots \lambda_n$. Now $n_i \leqq n$ and $\lambda_i \leqq M_i$ for every i, and M_i is obviously less than, or equal to, the Hadamard M. Hence $E \leqq n^{n/2} M^n$. If D has a zero-element or if there is an i for which $M_i < M$, the inequality sign, $<$, is effective.

SOME FUNCTIONS ANALOGOUS TO TRIGONOMETRIC FUNCTIONS

By L. E. WARD, University of Iowa

The differential equation $(d^2u/dx^2) + u = 0$ has the two linearly independent solutions e^{ix} and e^{-ix}, which are combined linearly to form the trigonometric functions sin x and cos x. It is the purpose of this note to define analogous functions based on the differential equation

$$(d^3u/dx^3) + u = 0, \tag{1}$$

and to derive some properties of them, pointing out the corresponding properties of the trigonometric functions.

We choose as a basis these three linearly independent solutions of (1): $e^{\omega_1 x}$, $e^{\omega_2 x}$, and $e^{\omega_3 x}$, where $\omega_1 = -1$, $\omega_2 = e^{\pi i/3}$, and $\omega_3 = e^{-\pi i/3}$ are the three cube roots of negative one. One third the sum of these is a function analogous to cos x; it is

$$u_1(x) = \tfrac{1}{3}(e^{\omega_1 x} + e^{\omega_2 x} + e^{\omega_3 x}). \tag{2}$$

The functions $u_2(x)$ and $u_3(x)$, written below, are respectively the negative of the first derivative and the second derivative of $u_1(x)$.

$$u_2(x) = -\tfrac{1}{3}(\omega_1 e^{\omega_1 x} + \omega_2 e^{\omega_2 x} + \omega_3 e^{\omega_3 x}). \tag{3}$$

$$u_3(x) = \tfrac{1}{3}(\omega_1^2 e^{\omega_1 x} + \omega_2^2 e^{\omega_2 x} + \omega_3^2 e^{\omega_3 x}). \tag{4}$$

Evidently $(d/dx)u_3(x) = -u_1(x)$, and consequently no new functions are obtained by further differentiation. The three functions $u_1(x)$, $u_2(x)$, and $u_3(x)$ are linearly independent solutions of (1), and each is real for real values of x. They are given in terms of letters not involving complex numbers by

$$u_1(x) = \frac{1}{3}\left[e^{-x} + 2e^{x/2}\cos\left(\frac{x(3)^{1/2}}{2}\right)\right];$$

$$u_2(x) = \frac{1}{3}\left[e^{-x} - 2e^{x/2}\cos\left(\frac{\pi}{3} + \frac{x(3)^{1/2}}{2}\right)\right];$$

$$u_3(x) = \frac{1}{3}\left[e^{-x} - 2e^{x/2}\cos\left(-\frac{\pi}{3} + \frac{x(3)^{1/2}}{2}\right)\right].$$

These formulas show that the functions are not bounded as $x\to\infty$, but that they do oscillate. They are not periodic.

The functions satisfy the boundary conditions

$$\begin{aligned} &u_1(0) = 1 \qquad u_2(0) = 0 \qquad u_3(0) = 0 \\ &u_1'(0) = 0 \qquad u_2'(0) = 0 \qquad u_3'(0) = -1 \\ &u_1''(0) = 0 \qquad u_2''(0) = 1 \qquad u_3''(0) = 0, \end{aligned} \tag{5}$$

and these together with the differential equation (1) determine the functions uniquely. The following power series in x may be obtained for the functions from (1) and (5), or from (2), (3), and (4) using the series for the exponential function:

$$u_1(x) = 1 - \frac{x^3}{3!} + \frac{x^6}{6!} - \cdots; \quad u_2(x) = \frac{x^2}{2!} - \frac{x^5}{5!} + \cdots;$$

$$u_3(x) = -\frac{x}{1!} + \frac{x^4}{4!} - \cdots.$$

They are series which are well adapted to computation if x is not too large numerically.

To derive a relation analogous to $\sin^2 x+\cos^2 x=1$, consider the function $\phi(x)=u_1^3(x)+u_2^3(x)+u_3^3(x)-3u_1(x)u_2(x)u_3(x)$. Differentiating with respect to x we find

$$\begin{aligned}\phi'(x) = &-3u_1^2(x)u_2(x) - 3u_2^2(x)u_3(x) - 3u_3^2(x)u_1(x) \\ &+ 3u_2^2(x)u_3(x) + 3u_3^2(x)u_1(x) + 3u_1^2(x)u_2(x) = 0.\end{aligned}$$

Hence $\phi(x)$ is constant and, taking $x=0$, we find the value of this constant to be unity. This gives the identity

$$u_1^3(x) + u_2^3(x) + u_3^3(x) - 3u_1(x)u_2(x)u_3(x) = 1.$$

The three functions possess simple addition theorems, namely

$$u_1(x+y) = u_1(x)u_1(y) + u_2(x)u_3(y) + u_3(x)u_2(y), \tag{6}$$

$$u_2(x+y) = u_2(x)u_1(y) + u_3(x)u_3(y) + u_1(x)u_2(y), \tag{7}$$

$$u_3(x+y) = u_3(x)u_1(y) + u_1(x)u_3(y) + u_2(x)u_2(y). \tag{8}$$

Of these, (7) and (8) are derived from (6) by differentiating with respect to either x or y, while (6) is found from the power series or from the definitions (2), (3), and (4).

Replacing y in (6), (7), and (8) by x, we get the formulas for the double variable,

$$u_1(2x) = u_1^2(x) + 2u_2(x)u_3(x) ; \qquad u_2(2x) = 2u_1(x)u_2(x) + u_3^2(x) ;$$
$$u_3(2x) = 2u_1(x)u_3(x) + u_2^2(x).$$

The formulas for the functions of the half variable can be found by solving these three equations for $u_1(x)$, $u_2(x)$, and $u_3(x)$. Although of degree six, the system can be solved in terms of radicals.

The relations $u_1(-\omega_3 x) = u_1(-\omega_2 x) = u_1(x)$ are analogous to $\cos(-x) = \cos x$, and are derived from (2). We have also $u_2(-\omega_3 x) = -\omega_2 u_2(x)$; $u_2(-\omega_2 x) = -\omega_3 u_2(x)$; $u_3(-\omega_3 x) = -\omega_3 u_3(x)$; $u_3(-\omega_2 x) = -\omega_2 u_3(x)$.

Of course, the list may be extended indefinitely, but the above will serve to indicate some of the resemblances of these functions to the trigonometric functions. We may also extend the work to a similar system of functions satisfying the differential equation $u^{(n)} + u = 0$.

RULED SURFACES REFERRED TO THE TRIHEDRAL OF A DIRECTRIX

By MALCOLM FOSTER, Williams College

Introduction. Let C be a curved line upon a ruled surface S, and let l be the ruling through a point M of C. Relative to the trihedral of C at M the coordinates of any point P on l are

$$x = \alpha t, \qquad y = \beta t, \qquad z = \gamma t, \tag{1}$$

where α, β, γ, are the direction-cosines of l expressed as functions of the arc s of C, and t is the distance along l from M to P.

1. The line of striction. Let l_1 be the ruling through a neighboring point M_1 on C. The direction-cosines of l_1 are $\alpha+\delta\alpha, \beta+\delta\beta, \gamma+\delta\gamma$, where[1]

$$\delta\alpha = d\alpha - \frac{\beta}{\rho}ds, \quad \delta\beta = d\beta + \frac{\alpha}{\rho}ds + \frac{\gamma}{\tau}ds, \quad \delta\gamma = d\gamma - \frac{\beta}{\tau}ds. \tag{2}$$

As M is displaced along C to M_1, any point P on l receives the following absolute displacements in the direction of the axes of the moving trihedral:[2]

[1] Eisenhart, *Differential Geometry of Curves and Surfaces*, p. 32.

[2] Eisenhart, p. 32.

$$(3)\qquad \delta x = dx + ds - \frac{y}{\rho}ds, \quad \delta y = dy + \frac{x}{\rho}ds + \frac{z}{\tau}ds, \quad \delta z = dz - \frac{y}{\tau}ds.$$

The condition that the locus of some point P be the line of striction is that the displacement of P be orthogonal to both l and l_1. Hence

$$\sum(\alpha + \delta\alpha)\frac{\delta x}{ds} = 0, \quad \sum\alpha\frac{\delta x}{ds} = 0.$$

Subtracting these equations and dividing through by ds we get

$$\sum\frac{\delta\alpha}{ds}\frac{\delta x}{ds} = 0, \quad \text{which becomes, on using (1), (2), and (3),}$$

$$\left(\alpha' - \frac{\beta}{\rho}\right)\left(\alpha t' + \alpha' t + 1 - \frac{\beta t}{\rho}\left(+\right)\beta' + \frac{\alpha}{\rho} + \frac{\gamma}{\tau}\right)\left(\beta t' + \beta' t + \frac{\alpha t}{\rho} + \frac{\gamma t}{\tau}\right)$$
$$+ \left(\gamma' - \frac{\beta}{\tau}\right)\left(\gamma t' + \gamma' t - \frac{\beta t}{\tau}\right) = 0,$$

where the primes indicate differentiation with respect to s.
Solving for t, we get the distance along l from M to the line of striction to be

$$(4)\qquad t = \frac{\dfrac{\beta}{\rho} - \alpha'}{\sum\alpha'^2 + \dfrac{2}{\rho}(\alpha\beta' - \alpha'\beta) + \dfrac{2}{\tau}(\beta'\gamma - \beta\gamma') + \dfrac{1}{\rho^2}(1 - \gamma^2) + \dfrac{1}{\tau^2}(1 - \alpha^2) + \dfrac{2\alpha\gamma}{\rho\tau}}$$

Since the condition that C be a geodesic is that $\beta = 0$, we have from (4) the well known theorem of Bonnet:

If a curve upon a ruled surface have two of the following properties, it has the third also:

1. *That it be a geodesic.*
2. *That it cut the rulings under constant angle.*
3. *That it be the line of striction.*

2. Developable surfaces. For l to generate a developable surface, the displacement of some point on l must be in the direction of the ruling. Hence for that point,

$$\frac{\delta x}{\alpha} = \frac{\delta y}{\beta} = \frac{\delta z}{\gamma}, \text{ or } \alpha\delta z - \gamma\delta x = 0, \quad \beta\delta z - \gamma\delta y = 0.$$

Using (3), these equations become

$$t\left(\alpha\gamma' - \alpha'\gamma - \frac{\alpha\beta}{\tau} + \frac{\beta\gamma}{\rho}\right) = \gamma, \quad t\left(\beta'\gamma - \beta\gamma' + \frac{\gamma\alpha}{\rho} + \frac{1 - \alpha^2}{\tau}\right) = 0.$$

Hence the condition that S be developable is that

$$(5)\quad t = \frac{\gamma}{(\alpha\gamma' - \alpha'\gamma - \alpha\beta/\tau + \beta\gamma/\rho)}, \quad \beta'\gamma - \beta\gamma' + \gamma\alpha/\rho + (1-\alpha^2)/\tau = 0,$$

where t is the distance from M to the edge of regression.

3. Condition that the directrix be a line of curvature. The tangent plane to S at any point M of C is determined by the ruling l and the tangent to C. Hence the normal to the surface is perpendicular to l and to the x-axis of the trihedral. The direction-cosines α_1, β_1, γ_1, of the normal are therefore determined from the following three equations:

$$\alpha_1 = 0, \quad \alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1 = 0, \quad \alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1.$$

We find

$$(6)\qquad \alpha_1 = 0, \quad \beta_1 = \gamma(\beta^2+\gamma^2)^{-1/2}, \quad \gamma_1 = -\beta(\beta^2+\gamma^2)^{-1/2}.$$

The condition that C be a line of curvature is that the ruled surface generated by the normal be developable. This condition will be given by substituting the values of α_1, β_1, γ_1, from (6) in (5). This gives after some reduction,

$$(7)\qquad t = \rho(\beta^2+\gamma^2)^{1/2}/\gamma, \quad \gamma\beta' - \beta\gamma' + (1-\alpha^2)\tau^{-1} = 0.$$

4. Orthogonal trajectories of the rulings. Let the point M be displaced along C to a neighboring point M_1. The condition that some point P be displaced orthogonally to the ruling is

$\sum \alpha \frac{\delta x}{ds} = 0$, which readily reduces to $t' + a = 0$ on using (1) and (3).

Hence
$$t = -\int a\,ds = \text{const.}$$

The point on C from which the arc s is measured may evidently be chosen so as to make the constant of integration equal to zero. Hence the distance t to a particular orthogonal trajectory may be given by

$t = -\int \alpha\,ds$, from which we have the following theorem:

If the rulings cut the directrix under a constant angle, the distance along a ruling from the directrix to an orthogonal trajectory is proportional to the arc of C; and if the directrix be a closed curve every orthogonal trajectory intercepts constant distances on each ruling.

5. Locus of the center of geodesic curvature of an orthogonal trajectory of the rulings. Let the directrix C be an orthogonal trajectory of the rulings. It is well known that the tangent planes at M and P, the center of geodesic

curvature of C, are at right angles. Hence the direction-cosines of the normal to the surface at P are (1, 0, 0). We consider the ruled surface generated by these normals. The coordinates of any point on this normal are

$$x = t, \quad y = \rho, \quad z = \rho\gamma/\beta, \tag{8}$$

where t is the distance from P. We have seen that the distance to the line of striction is determined by the equation

$$\sum \frac{\delta\alpha}{ds}\frac{\delta x}{ds} = 0, \quad \text{which readily reduces to} \quad t = -\rho\left(\rho' + \frac{\rho\gamma}{\beta\tau}\right).$$

Since a characteristic property of asymptotic lines is that they are the lines of striction of ruled surfaces formed by the normals along these lines, the last equation tells us that $\rho' + (\rho\gamma/\beta\tau) = 0$ is the condition that the locus of P be an asymptotic line. From this we readily get $\rho = ce^{-\int \gamma ds/\beta\tau}$.
Hence if the curve C be plane, it is also a circle.

We now consider the condition that the locus of P be a line of curvature. This condition is that the normal at P generate a developable surface. Hence for some point on this normal $\delta y = 0$, $\delta z = 0$. From (8) these two equations become

$$t = -\rho\left(\rho' + \frac{\rho\gamma}{\beta\tau}\right), \quad \rho\beta' - \gamma^2\beta\rho' + \frac{\gamma\beta^2\rho}{\tau} = 0,$$

where t is the distance to the edge of regression.

The condition that the locus of P be the line of striction is that t as given by (4), satisfy the equation $t = \rho/\beta$. This reduces to

$$\frac{\dfrac{\beta}{\rho}}{\left(\dfrac{\beta'}{\gamma} + \dfrac{1}{\tau}\right)^2 + \dfrac{\beta^2}{\rho^2}} = \frac{\rho}{\beta}, \quad \text{from which} \quad \frac{\beta'}{\gamma} + \frac{1}{\tau} = 0.$$

Since $\alpha = 0$, we may write this in the form $\beta'(1-\beta^2)^{-1/2} + \tau^{-1} = 0$. Hence arc cos $\beta = \int \tau^{-1} ds +$ const., is the desired condition. From this we have the theorem:

If C be a plane curve, the condition that P describe the line of striction is that $\beta = const.$; and if C be a curve of constant torsion, the angle between the principal normal and the ruling is proportional to the arc of C.

6. The line of striction an asymptotic line. Let C be the line of striction then, from (4), $\beta = \alpha'\rho$. If C be also an asymptotic line, $\gamma = 0$, and the relation

$\sum\alpha^2=1$ becomes $\alpha^2+(\alpha')^2\rho^2=1$. This may be put in the form $\rho^{-1}=\alpha'(1-\alpha^2)^{-1/2}$, from which we get arc $\cos\alpha=-\int\rho^{-1}ds+\text{const.}$, from which we have the theorem:

On a ruled surface whose line of striction is an asymptotic line of constant first curvature, the rulings meet the line of striction at an angle which is proportional to the arc.

7. The rulings the characteristics of the tangent planes along C. From (6), the equation of the tangent plane at M is $\gamma y-\beta z=0$. The characteristic of the tangent plane as M is displaced along C will be defined by this equation and[1]

$$\gamma\left(\frac{x}{\rho}+\frac{z}{\tau}\right)+\frac{\beta y}{\tau}-y\gamma'+z\beta'=0.$$

If we eliminate y and z between the last two equations, we get

$$\frac{\gamma^2x}{\rho}-kz=0, \quad \frac{\beta\gamma x}{\rho}-ky=0,$$

where we have set $k=\beta\gamma'-\gamma\beta'-\tau^{-1}(1-\alpha^2)$. From (9), $x:y:z=\rho k:\beta\gamma:\gamma^2$. Hence the direction-cosines of the characteristic are

$$(10)\quad \alpha_1=\frac{\rho k}{[\rho^2k^2+\gamma^2(1-\alpha^2)]^{\frac{1}{2}}},\quad \beta_1=\frac{\beta\gamma}{[\rho^2k^2+\gamma^2(1-\alpha^2)]^{\frac{1}{2}}},\quad \gamma_1=\frac{\gamma^2}{[\rho^2k^2+\gamma^2(1-\alpha^2)]^{\frac{1}{2}}}.$$

We seek the condition that the rulings be the characteristics. We must have

$$\rho k/[\rho^2k^2+\gamma^2(1-\alpha^2)]^{1/2}=\alpha \text{ from which we get } (\rho^2k^2-\alpha^2\gamma^2)(1-\alpha^2)=0.$$

Hence, excluding the case where $1-\alpha^2=0$, which is the condition that S be developable, the above becomes

$$(11)\qquad (\rho k+\alpha\gamma)(\rho k-\alpha\gamma)=0.$$

When (11) holds, it is readily seen from (10) that $\beta_1=\beta$ and $\gamma_1=\gamma$. Hence (11) is the condition that the rulings be the characteristics. If k vanishes we must have $\alpha=0$.[2] But if k is zero, we note from (7) that C is a line of curvature. This is, of course, what we should expect as the lines of curvature form the only orthogonal conjugate system.

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[1] Eisenhart, p. 65.

[2] We exclude the case where $\gamma=0$, that is, where C is an asymptotic line.

DID THE ARABS KNOW THE ABACUS?

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The modern system of numeration with place value and zero certainly belongs among the greatest inventions of the human spirit. "Who it was to whom the invention is due, or where he lived, or even in what country, will probably always remain a mystery."[1] So far, however, we know that this system is closely connected with the so-called Hindu-Arabic numerals and that its origin therefore is to be traced to Hindu and Arabic sources.[2]

On the other hand, we must not imagine that these Hindu-Arabic numerals, with their ingenious system of place value and zero, owe their inception to a spontaneous idea suddenly originating in the mind of some great mathematician. For many hundreds of years the system of place value has been practiced in a mechanical and palpable way on the abacus in its various forms. On the columns of the abacus the Hindus most probably developed their famous system of notation with a special name for each power of ten.[3] On the columns of the abacus the Hindu mathematicians probably first learned to connect different orders of the decimal system with the same marks, counters, or symbols. It seems not only possible but very likely "that one of the forms of ancient abacus suggested to some Hindu mathematician the use of a symbol to stand for the vacant line when the counters were removed."[4] Bubnov and Kaye, although denying the Hindu origin of the numerals, still admit "that the forms of the numerals first found in Europe are derived from ancient symbols used on the abacus."[5] The Hindu-Arabic numerals were not improperly characterized as a means to transform the instrumental abacus into a written abacus;[6] and indeed they are nothing else, the columns being missing but being inserted mentally.

We therefore may find it quite natural that India, which is "historically the home of the Abacus," is at the same time the birthplace of the Hindu numerals representing the written abacus.[7] We are, however, naturally sur-

[1] Smith-Karpinski, *The Hindu-Arabic numerals*, p. 41.

[2] Smith-Karpinski, loc. cit., pp. 47 seq., 91 seq., 112 seq., 117 seq.

[3] Alberuni, *India*, English ed., vol. 1, pp. 174 seq.; Woepcke, *Mémoire sur la propagation des chiffres Indiennes*, Paris, 1863, pp. 92 seq., 110 seq.; Smith-Karpinski, loc. cit., pp. 41 seq.

[4] Smith-Karpinski, loc. cit., p. 41.

[5] Smith-Karpinski, loc. cit., p. 65; Kaye in the Journal of the Asiatic Society of Bengal (1907), p. 488.

[6] Woepcke, *Mémoire sur la propagation*, pp. 67 and 128.

[7] The existence of an abacus in India and the close connection of the Hindu numerals with the abacus may be regarded as the common opinion of the historians of mathematics and Indian literature; see Humboldt, *Kosmos*, vol. 2, pp. 263 seq., 454–456; Chasles, quoted by Humboldt, *ibid.*, p. 454, note 18; Woepcke, loc. cit.; Knott, "The Abacus," in The Transactions of the Asiatic Society of Japan, vol. 14, pp. 19–20, says: "The modern cipher system of notation . . . . owes its origin to the indications

prised at the strange fact that the Arabs who were the first to accept from India this system of the written abacus, namely the Hindu numerals, and to improve it, and to teach and spread it all over the world, should have known nothing of its origin, the instrumental abacus. It is therefore the purpose of this paper to examine once more the old question, "Did the Arabs know the abacus?"

This problem has for a long time been a matter of controversy among historians of mathematics and is not yet solved, and a brief sketch of the points at issue will be appropriate as an introduction to the writer's own theory.

Woepcke[1] thinks it very likely that the Arabs knew the column-abacus. He says: "Tous ces faits rendent plus que probable que les Arabes, en arrivant en Espagne, y adoptèrent pareillement les chiffres et .... le tableau à colonnes." The German mathematicians Hankel and Treutlein strongly object to this opinion. Hankel[2] contends: "Jene alten [*ghobâr*] Ziffern erscheinen bei den Lateinern so wesentlich mit dem Abakus verbunden, ... dass man meinen sollte, der Abakus müsse aus derselben Quelle stammen, wie jene Ziffern. Nun aber findet man in der gesamten arabischen Literatur nicht die geringste Spur einer Bekanntschaft mit dem Abakus." This means: ***There is not the slightest trace of an acquaintance with the abacus in the whole Arabic***

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of the Abacus." "The Abacus finds its earliest historic home in India." Further on, p. 44, Knott continues as follows: "Historically, the home of the Abacus is in India. .... Once the decimal stage was reached, its general similarity to the Abacus indications suggested bringing them into still closer correspondence. This advance seems to have taken place amongst the Aryan Indians, who .... very soon discarded the Abacus for the more convenient cipher notation." This opinion is also accepted by Smith, *History of Mathematics*, vol. 2, pp. 158 and 159, and by Smith-Karpinski, loc. cit., p. 65. Of the same opinion are also Bayley, Taylor, Woepcke, Burnell and Rodet, all quoted by Kaye, "*The use of the abacus in ancient India*," in the Journal of the Asiatic Society of Bengal, 1908, pp. 293–298, and also *ibid.*, 1907, pp. 119, 487 seq. See also Fleet, "*The use of the abacus in India*," in the Journal of the Royal Asiatic Society, 1911, who interprets the word *ganitra* occurring in a writing of the first century of the Christian era as meaning an instrument of reckoning. Kaye himself (*ibid.*, 1907, p. 488; 1908, p. 297) says "that the abacus played a very important part in the development of our notation no one will deny," but he questions the fact that it was known to the Hindus on the ground that "if such an instrument as the abacus were in use in ancient India some real evidence of its use would be forthcoming." That such evidence does not exist is also asserted by the Hindu scholar Datta in this Monthly (1926), p. 450. Datta, however, who also refers to Fleet's theory, mentioned above, formulates his opinion as follows: "Until now no mention of the existence of the abacus, direct or indirect, has been traced in any Indian literature so that it may be taken without any fear of contradiction that if an abacus was ever in use among the learned men of India, it was discarded long ago." However, Cantor (*Geschichte der Mathematik*, vol. 1, 4th ed., p. 609) simply denies that there is any proof whatever for the existence of the abacus in that country. That Cantor is consistent, insofar as he also denies that the Arabs knew the abacus, is considered later in this paper.

[1] *Mémoire sur la propagation* (Paris, 1863), p. 58.

[2] *Geschichte der Mathematik* (1874), p. 328.

literature.[1] In the same way Treutlein expresses his opinion[2] "dass die Araber unzweifelhaft kein Columnenrechnen hatten"; that is, that undoubtedly the Arabs had no column-arithmetic. Weissenborn first takes exception to these statements of Hankel and Treutlein, which were made with such a certainty, and admits the possibility of the Arabs having known the abacus. He says:[3] "Da vernahm man die Araber rechneten auf einer mit Staub bedeckten Tafel .... und was konnte eine solche Tafel, welche zum Rechnen diente, anders sein als ein Rechenbrett?" And then he continues:[4] "Allein sie rechneten doch auf einer mit Staub bedeckten Tafel, freilich nicht mit Marken, sondern indem sie die Zahlzeichen schrieben. Ob nun unter solchen Umständen diese Staubtafel ein 'Abakus' genannt werden kann, darüber wird man streiten können." Then he quotes the testimony of the English chronicler William of Malmesbury (*c.* 1150), that Gerbert learned the use of the abacus from the Arabs[5] and, still doubting, he concludes in his summary[6] with the admission of two possibilities: Either Gerbert invented the abacus, or he learned it from the Arabs. But it was only an abacus with written statements "Schriftliches Verfahren," namely the dust abacus with the dust numerals. Later on, Weissenborn with still more emphasis ascribes the knowledge of the abacus to the Arabs, saying:[7] "It is true that we do not find the abacus mentioned among the Arabs, but we must not draw the conclusion that it was not known to them." His reasons are the testimony of William of Malmesbury mentioned above, and a treatise of Al-Kindi (*c.* 850), —"*De numeris per lineas et grana hordeacea multipl candis.*"[8] Finally Weissenborn refers to the inner,

[1] According to Kaye, in the Journal of the Asiatic Society of Bengal (1907), p. 488, Rodet is supposed to have stated that in Arabic and Persian manuscripts one often comes across arithmetical calculations effected by means of the "tableau à colonnes." Kaye does not quote exactly the passage where Rodet made such a statement. It is to be found in the Journal Asiatique, (7), vol. 16 (1880), p. 463.

[2] *Geschichte unserer Zahlzeichen* (1875), p. 36.

[3] *Gerbert* (Berlin, 1888), p. 230.

[4] *Ibid.*, p. 235.

[5] *Ibid.*, p. 236: "Abacum certe primus a Saracenis rapiens regulas dedit"; see also Smith, *History*, vol. 2, p. 175.

[6] *Gerbert*, pp. 237–239.

[7] *Zur Geschichte der Einführung der jetztigen Ziffern in Europa* (Berlin, 1892), pp. 5 seq.

[8] This Latin translation is given by Casiri, *Bibliotheca Arabico Hispana*, vol. 1, pp. 353 seq. The Arabic title, however, reads: "*Risâlat fî'l khuṭûṭ w'al ḍarb bi'a'dâd al-sha'ir*" (see Casiri, *ibid.*, p. 357; *Fihrist*, Arabic text, p. 256, line 22; *Ibn Abi Useibia*, ed. Müller, vol. 1, p. 210, line 15), and ought to be translated "A treatise on the lines and the multiplication with the number of the barley-grains." This translation is also given by Suter (*Fihrist*, p. 11), and accepted by Cantor (vol. 1, 2nd ed., p. 675; 4th ed., p. 718) in his polemics against Weissenborn's theory. On the other hand this correct translation gives no sense. The writer therefore prefers the Latin translation of Casiri, which clearly proves the knowledge of the line abacus, and is the work of one who was thoroughly familiar with Arabic and who made no mistake in his translation. His rendition of the text indicates that he could find no sense in the extant Arabic title, and believed that it should read "*Risâlat fî ḍarb al'a'dâd bi'l khuṭûṭ w'al-sha'ir*," to which emendation the writer would also subscribe. For further literature on Al-Kindi see Steinschneider, *Hebräische Uebersetzungen*, pp. 562 seq., and note 174 *ibid.*

historical evidence, saying:[1] Es war mir schon früher nicht glaubhaft, dass die Araber (sie hätten ja sonst wohl unter allen Völkern allein gestanden) vom Kopf- und Fingerrechnen unmittelbar zum schriftlichen Rechnen übergegangen sein sollten, ohne, wenigstens eine kurze Zeit hindurch, ein Rechenbrett zu benutzen." Cantor, however, opposes Weissenborn's theory with great determination, although without mentioning his name, saying:[2] "Von einem Rechenbrette oder etwas, was demselben irgendwie gleicht, ist bei Alchwarismi keine Rede, und ebenso erfolglos wird unser Suchen danach bei allen arabischen Schriftstellern bleiben." He adds: "Von Al-Kindi wird zwar eine Schrift über die Linien und das Multiplizieren mit der Zahl der Gerstenkörner erwähnt, aber daraus auf einen Abacus zu schliessen, dürfte allzukühn sein"; and further on we read:[3] "Von Arabern, bei welchen die Kunst [des Abacus] geblüht haben könnte, ist auch bei Radulph von Laon (*d.* 1131) mit keinem Worte die Rede . . . . Auch von Atelhart von Bath, welcher um 1130 über den Abakus schrieb, ist ein sehr beredtes Schweigen zu melden." Atelhart, although he perfectly mastered the Arabic language, as is shown by his translations, did not mention the Arabs in his treatise *Regulae Abaci*. Sein [Radulph von Laon's] Schweigen [about the Arabs; he ascribes the abacus to the Assyrians] ist als Beweis anzusehen, dass ihm und mit ihm den Zeitgenossen, vor welchen er durch Gelehrsamkeit sich auszeichnete, ein Vorkommen des Abakus bei den Arabern gerade so unbekannt war wie bei uns." Further on we read:[4] "Wesentlich ist auch das Nichtvorkommen des Abacus im Algorithmus des Johannes von Sevilla." And finally he says:[5] "Das Kolumnenrechnen kommt erst bei Ibn al-Bannâ am Ende des dreizehnten Jahrhunderts vor; hier ist abendländischer Einfluss nachgewiesen."

From the English school, however, Professors D. E. Smith and L. C. Karpinski seem fully to accept Weissenborn's theory, and, without further entering into the discussion of the problem, are simply stating[6] that "the significance of the term *ghobâr* is doubtless that the numerals were written on the dust-abacus." Still more clearly Professor Smith gives as his opinion[7] that the dust-board was common among the Arabs in the Middle Ages, and the numeral forms derived from being written on such a tablet were therefore called in the schools of the western Arabs the ghobâr numerals. Not satisfied with this Professor Smith started an investigation relating to the existence of the ghobâr numerals among the eastern Arabs, concerning which he also enter-

[1] *Zur Geschichte*, as above, p. 8.
[2] *Geschichte der Mathematik*, vol. 1, 2nd ed., p. 675; 4th ed., p. 718.
[3] *Ibid.*, vol. 1, 2nd ed., pp. 836, 838; 4th ed., pp. 891, 893.
[4] *Ibid.*, vol. 1, 2nd ed., p. 854; 4th ed., p. 909.
[5] *Ibid.*, vol. 1, 2nd ed., p. 758; 4th ed., p. 807.
[6] Smith-Karpinski, *The Hindu-Arabic Numerals*, p. 65.
[7] *History*, vol. 2, p. 175.

tained a correspondence with his friend Salih Mourad, then a professor in the Turkish Naval College, Constantinople. Finding then that the eastern Arabic scholars Al-Antâkî and Al-Kalwâdânî, living in the tenth century in Bagdad, wrote certain books under the title "The book of the board on Hindu arithmetic," both Smith and Salih Mourad inferred that the eastern Arabs used the dust numerals in the same way as the western computers of Cordova, Salamanca and Toledo. Professor Smith was then kind enough to show to the writer of these lines the incomplete manuscript prepared by them upon the subject, giving him (with Professor Salih Mourad's approval) liberty to use it.

While studying these notes and trying to collect some further material of the same kind the writer was struck by the idea that there is indeed abundant proof and clear evidence for the statement that the eastern as well as the western Arabs knew not merely the dust abacus but the general form as well, and created two Arabic terms for abacus, which have not as yet been recognized as such by students of Arabic philology and mathematics. This material, together with the writer's opinion, is now submitted as follows:

'Alî ibn Aḥmed, Abû'l Qâsim, al-Antâkî (that is from Antioch), al-Mujtabâ (the chosen), who lived at Bagdad and died in 987, wrote several books on arithmetic bearing the name *Kitâb al-takht* "The book of the board."[1] They are (1) The *Kitâb al-takht al-kabîr fî'l ḥisâb al-Hindî*, "The great book of the board on Hindu arithmetic"; (2) *Kitâb fî'l ḥisâb 'alâ'l takht bilâ maḥw*, "The book on arithmetic on the board without erasing"; (3) *Kitâb al-ḥisâb bilâ takht bal bi'l yad*, "The book on arithmetic with the hand without the board."[2] Mohamed ibn Abdallâh, Abû Nasr, al-Kalwâdânî, a contemporary of al-Mujtabâ and, like him, living at Bagdad, also wrote a book entitled *Kitâb al-takht fî'l ḥisâb al-Hindî*, "The book of the board on Hindu arithmetic."[3] Sinâ ibn al-Fatḥ (whose father, Al-Fatḥ, may be identical with Fatḥ ibn Naghiyya (died *c.* 941), a native of Harrân and a mathematician of some merit) is also mentioned as the author of a book *'Ilm ḥisâb al-takht*, "The science of arithmetic of the board."[4] There is also a *Kitâb al-takht* written by al-Râzî, whose date is unknown.[5]

[1] See *Fihrist*, pp. 40 and 75, nn. 271, 276; Arabic text of the *Fihrist*, p. 284; Suter's list, p. 63; Casiri, loc. cit., vol. 1, p. 411 (quoted also by Weissenborn, *Zur Geschichte*, p. 87); Ibn al-Qifṭi, *Tarikh al-ḥukamâ*, ed. Lippert (Leipzig, 1903), p. 234; Woepcke, *La propagation*, pp. 157, 160. In the library of the Jewish Theological Seminary there is a copy of Woepcke, formerly in the Steinschneider library, which contains on the above-mentioned pages numerous marginal notes by Steinschneider referring to further literature upon the subject.

[2] This book is quoted directly by Ibn al-Qifṭi, and by others on his authority.

[3] *Fihrist*, pp. 41, 75; Arabic text, vol. 1, p. 284; Suter's list, p. 74; Ibn al-Qifṭi, loc. cit., p. 288; Woepcke, *loc. cit.*

[4] *Fihrist*, pp. 37, 70; Arabic text, vol. 1, p. 281; Suter's list, pp. 51, 66; Woepcke, loc. cit., p. 157.

[5] *Fihrist*, p. 37; Arabic text, p. 281 at the bottom.

The question now arises as to what is the exact meaning of *Kitâb al-takht*, "Book of the board."[1] It can hardly be assumed that a kind of a board tablet of the Arabic school is meant. There is nothing characteristic on such a board to give the name to the science of arithmetic. On the other hand we see that all the treatises named *Kitâb al-takht* dealt with the Hindu arithmetic. Now we know that the characteristic of the Hindu numerals was the place value and the zero. This was also the characteristic of the abacus. It would therefore be proper to call the Hindu arithmetic the abacus arithmetic. Besides that we know that the abacus gave the name to arithmetic in general, like the calculus (pebble).[2] We know that "the period from the time of Gerbert (*c.* 1000) until after the appearance of Leonardo's monumental work (1202) is called the period of the abacists."[3] But although "even for many years after the appearance, early in the twelfth century, of the books explaining the Hindu art of reckoning, there was strife between the abacists, the advocates of the abacus, and the algorists, those who favored the new Hindu numerals,"[4] the word abacus, however, finally came to mean any kind of elementary arithmetic, and even books dealing with the algorismus and the Hindu-Arabic numerals bore the name *Liber Abaci* or *Libro d'abacho*,[5] and even the famous book of Leonardo Fibonacci, "which was most influential in introducing the new algoristic method based on the Hindu-Arabic numerals to the scholars of Europe,"[6] bore the name *Liber Abaci*.

Taking into consideration all these facts the writer is convinced that the word *takht* ("board") is nothing else than the Persian-Arabic term for the abacus. The *Kitâb al-takht* is the Arabic *Liber Abaci*,[7] and the authors of

[1] The form *takht* should be used instead of *taḫt*, although the *Fihrist* and Hâji Khalfa (see hereinafter) adopted the latter. The *ḫ* is the Arabic letter *ḥ* without a dot, whereas *kh* represents the same letter with a dot and is more guttural than the other. *Takht* is the more correct form and is therefore properly used by Woepcke, Ahlwardt (mentioned later), and Ibn al-Qifṭi. Suter first erroneously read *taḫt* and translated "method," but soon recognized that *takht* = "board" is the better reading; see his *Fihrist*, p. 70, note 232. *Takht* is merely an Arabic form of the Persian word *takhta* meaning "board" or "wood"; see Vuller, *Lexicon Persico-Latinum*, vol. 1, p. 425; Lane, *Arabic Lexicon*, p. 298.

[2] Smith, *History*, vol. 2, 166.

[3] Smith-Karpinski, p. 120; Cantor, vol. 1, chapter 40 on the abacists and algorists.

[4] Smith-Karpinski, *loc. cit.*

[5] Smith, *History*, vol. 2, p. 7; Cantor, vol. 1, chapter 40.

[6] Smith-Karpinski, pp. 129 seq.

[7] Leonardo Fibonacci, "the most noteworthy mathematical genius of the Middle Ages, ... the one most influential in introducing the new numerals to the scholars of Europe" (Smith-Karpinski, p. 129), was a disciple of the Arabs, and his *Liber Abaci* is only "eine freie Darstellung" (a free presentation) of the entire knowledge of the Arabs about arithmetic and algebra; see Hankel, loc. cit., pp. 293, 342. As we now see, the very title *Liber Abaci*, which was so common in the period of the abacists, is only a translation of the Arabic *Kitâb al-takht*. In the same way the title of another work of Fibonacci, *Practica Geometriae*, is also only a literal translation of the Arabic book titles *Al-Ṣinâ'a al-Handasiyya* and *Ṣinâ'at al-handasa* (cited by Woepcke, *La propagation*, p. 178, and by Ahlwardt, *Catalogue of Arabic manuscripts*, Berlin, vol. 5, p. 321, No. 5945 in fine). The more correct translation is *Ars Geometriae*,

those treatises are the Arabic abacists, the fore-runners of the European abacists. The book of Al-Mujtabâ was not the first of its kind, for he called it the great *Kitâb al-takht*. This indicates that there were already numerous such treatises, and that the name *Kitâb al-takht* was common and popular for works of this kind. Al-Mujtabâ undertook to write a compendious arithmetic which he called "The great *Liber Abaci*."[1]

Further evidence for the identity of the *takht* and the abacus may be found in the following book titles:[2] *Kâfiyya fî ḥisâb al-takht w'al-mîl*, "A compendium of arithmetic with board and stylus," written by Amîn al-Dîn, al-Abharî, who died in 1332; and the *Jawâmi' al-ḥisâb bi'l takht w'al-turâb*, "An encyclopedia of arithmetic with board and dust," written by Abû 'Abdallâh al-Zanâtî, whose date is unknown. Here we have all the requisites of the dust abacus, the tablet with dust and stylus.[3] Further corroboration is to be found in a note of Haji Khalfa,[4] which reads as follows: "There are many divisions of arithmetic. One of them is the science of arithmetic of the board and the stylus, *al-takht w'al-mîl*. This is the science by which we learn how to perform the arithmetic operations by numerical symbols indicating the units, and also numbers exceeding the units, by the stages.[5] These symbols are attributed to the Hindus. . . . . This science is also called 'The board and the dust,' *al-takht w'al-turâb*." Here it is clearly stated that there is a science of "the board and the dust," or "the board and the stylus," which deals with the Hindu numerals and the place value, and this admits no other conception than that of the science of abacus arithmetic.

But the Arabs not only knew the abacus, they also knew the double etymology of the Greek word *abakion*, namely as "table, tray," and as "dust,"

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sinâ'a meaning "art, handicraft." The *Liber Abaci* was also called by the Arabic name *Algebra et Almuchabala*; see Ball, *History of Mathematics*, 6th ed., p. 167. Further instances of the Arabic influence are evident in *regula elchatayn* and in the word *pensa* (weight, the Arabic *mizân*) for "proof"; see Cantor, vol. 2, 1st ed., p. 25; Tropfke, vol. 1, 2nd ed., p. 60. That Leonardo was under the influence of the Hebrew mathematicians Savasorda and Ibn Ezra has been satisfactorily proved, his *Practica Geometriae* being an imitation of the *Liber Embadorum* of Savasorda. The arrangement of the chapters is the same, and a great number of sentences and examples are verbally taken from the same work. On this point see Steinschneider, *Gesammelte Schriften*, pp. 405 seq., and in Orientalische Literatur Zeitung (1901), p. 93; Curtze, *Liber Embadorum* (in Abhandlungen zur Geschichte der Mathematik, 1902), p. 5. In his *Liber Abaci* Leonardo followed the same order and arrangement as in the arithmetics of Savasorda and Ibn Ezra; see Steinschneider, *Gesammelte Schriften*, p. 471. Woepcke, *Extrait du Fakhrî*, p. 3, proves that Fibonacci was also indebted to al-Karkhî.

[1] At the same time he expounded also the finger arithmetic which he called "The arithmetic with the hand without the abacus." Compare Heath, *Greek Mathematics*, vol. 1, p. 48: "In Aristophanes (*Wasps*, 656–64) Bdelycleon tells his father to do an easy sum *not with pebbles but with fingers*."

[2] See Ahlwardt, *Catalogue of Arabic manuscripts in Berlin*, vol. 5, p. 333, No. 5975; p. 352, No. 6013, 8.

[3] Or also the pointer; see Smith, *History*, vol. 2, pp. 157 seq., 177 seq.

[4] *Lexicon Bibliographicum et Encyclopaedicum*, the Arabic text with a Latin translation, by G. Flügel, in 7 volumes (Leipzig, 1835–1858). The note is in volume 3, pp. 61 seq.

[5] Or "by degrees," referring to place value.

from the Hebrew *abaq*.[1] While the eastern Arabs rendered it with the Persian term *takht*, "board, tray," the western Arabs called it *ghobâr*, "dust," and sometimes also *turâb*, which likewise means "dust." But this well-known term *ghobâr* does not mean "dust" or "dust-board," it is simply an attempt to arabicise the Greek term *abakion* (abacus). There is no more reason to restrict the *ghobâr* to the dust abacus alone than we have reason to restrict the Greek-Latin abacus to the dust abacus alone, because it is possibly derived from the Hebrew *abaq*. *Takht* and *ghobâr* are simply two Arabic terms for the translation of the word "abacus."[2] Now the word abacus may be derived from the ancient dust table, and may also, very probably, have another explanation. Certain it is that the word "abacus" was also applied to the counter- and line-abacus. Therefore the words *takht* and *ghobâr* must also have been used to signify these forms of the abacus.[3]

In making the assumption that the well-known *ghobâr* is nothing else than the technical term of the Arabs for the abacus, all the data and terms connected with the word *ghobâr* appear at once in a new light, fit more smoothly into the general history of mathematics, and throw more light on the data and terms used by Europeans.

We understand now why the *ḥisâb al-ghobâr* (abacus arithmetic) in Spain is like the *ḥisâb al-hindi* and the *ḥisab 'alâ'l takht* in the Orient.[4] We understand why the *ḥurûf al-ghobâr* (the abacus numerals) are identical with the symbols and characters of the medieval abacus.[5] We understand why there is a *ḥisâb al-ghobârî* (abacus arithmetic) in opposition to the *ḥisâb al-hawâî* (the mental

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[1] See the Greek dictionaries; Smith, *History*, vol. 2, p. 156; Cantor, vol. 1, 4th ed., p. 131.

[2] The abacus was also known to the Hebrews, and there are two Hebrew terms for it: *pinqas* (the Greek *pinax*), "board, tablet," and *abaq*, *aphar*, "dust," corresponding to the Arabic *takht* and *ghobâr*. The *ghobâr*- or dust-numerals are called in Hebrew *mispar ha-abaq* and *mispar ha-aphar*. Abu Sahl ibn Tamîm (*c.* 955), better known as Isaac Israeli, wrote a book *Ḥisâb al-ghobâr beḥeshbôn anshê hôdô*, "The abacus arithmetic on Hindu arithmetic." This strange title, half Arabic and half Hebrew, sounds like the above mentioned *Kitâb al-takht fî ḥisâb al-Hindi*. Naturally, these Hebrew terms for the abacus have not been recognized as such until now, but have been misunderstood like the terms *ghobâr* and *takht*. On these Hebrew terms and upon Isaac Israeli and his book see Steinschneider, *Jewish Literature* (London, 1857), p. 363; *Hebräische Uebersetzungen*, p. 397, note 196, and pp. 400, 558; M. Reinaud, *Mémoire . . . . sur l'Inde*, p. 399. Reinaud, from whose book the European mathematicians drew their knowledge of these passages, owed his information to M. S. Munk; see Munk, *Notice sur Abou'l Walid*, p. 51.

[3] The terms *takht w'al-mtl* and *takht w'al-turâb*, mentioned above, will be best explained as attempts to combine the two etymologies of the table and the dust-table. Hence these terms cannot be used as proof for the real existence of the dust-abacus.

[4] See Woepcke, *La propagation*, pp. 59, 61, 182 seq.

[5] See Weissenborn, *Gerbert*, pp. 227, 237–239; the same author, *Zur Geschichte*, pp. 10 seq.; Hankel, p. 328, quoted above; Smith-Karpinski, pp. 117 seq.; Smith, *History*, II, pp. 74 seq.

or head-arithmetic).[1] We understand the terms *'ilm al-ghobâr* and *'ilm al-turâb*[2] which are like the *'ilm al-takht*[3] (the science of the abacus).

As already mentioned above, the term *turâb* ("dust"), is also used, like *ghobâr*, to signify the abacus. This will help us to better understand the famous passage in Alberuni's (Al-Bîrûnî's) *India* dealing with the Hindu numerals. The passage reads:[4] "As in different parts of India the letters have different shapes, the numeral signs, too, ... differ. The numeral signs which we [the Arabs] use are derived from the finest form of the Hindu signs. . . . . But the people of Kashmîr mark the single leaves of their books with figures which look like drawings or like the Chinese characters. . . . . However, they do not use them when reckoning in the sand." From the last sentence the historians usually inferred that the Hindus were in the habit of reckoning on the sand. But the sentence was misunderstood by Sachau, the English translator, as well as by Woepcke.[5] The Arabic text in Sachau's edition[6] reads: *walâ tasta'malu fi'l-ḥisâb alâ'l-turâb*, meaning: "And they are not used in computing on the abacus." It refers to the Hindu-Arabic numerals mentioned above, and relates the well-known fact that the usual Hindu-Arabic numerals differ from the *ghobâr*- or abacus numerals.[7]

In view of all these facts it is hoped that historians will agree that the opinions of Hankel, Treutlein and Cantor to the effect that there is not the slightest trace of the abacus in the whole Arabic literature[8] will no longer be accepted, but that we shall recognize a new chapter in Arabic mathematics on the Arabic abacus and the Arabic treatises with the name *Liber Abaci*.[9]

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[1] Woepcke, "Recherches," Extrait no. 13 from the Journal Asiatique (1854), pp. 11 seq.

[2] Woepcke, *ibid.*; Ahlwardt, *loc. cit.*, vol. 5, pp. 336, 352, nn. 5979, 6013, 7.

[3] Haji Khalfa, *loc. cit.*

[4] Alberuni's *India*, English text, vol. 1, p. 174.

[5] *La propagation*, pp. 95 seq.

[6] P. 83, lines 2–3.

[7] See Smith-Karpinski, pp. 68 seq., 98, 112 seq., 140.

[8] Cantor himself admits (Vol. 1, 4th ed., p. 807, quoted above) that Ibn al-Banná knew the column-abacus, but he says that it is proved that there was European influence. The writer does not know how or where this is done. But he well knows that this treatise of Ibn al-Banná is only an extract of a treatise of Ibn al-Ḥaṣṣâr, who lived in the twelfth century, and that Ibn al-Ḥaṣṣâr expressly states that he gives nothing of his own, having collected his material from the old Arabic mathematicians; see Suter, "Das Rechenbuch des Abû Zakarîja el-Ḥaṣṣâr," Bibliotheca Mathematica, (2), vol. 2 (1901), pp. 12 seq., 40. So we have, on the contrary, evidence of an old and genuine Arabic influence. See also Rodet in Journal Asiatique (7), vol. 16 (1880) p. 469.

[9] The writer wishes to express his appreciation of the assistance given by Professor David Eugene Smith in the preparation of his manuscript.

THE TWENTIETH ANNIVERSARY OF "SCIENTIA"

By DAVID EUGENE SMITH, Teachers College, Columbia University

It seems appropriate that The American Mathematical Monthly should call attention at this time to the fact that twenty years have elapsed since the first appearance of the international publication *Scientia*, and to extend to its sponsors congratulations upon the success which it has achieved. Founded in 1907 by Signori Bruni, Dionisi, Enriques, Giardina, and Rignano, it was published under their direction until 1925, since which time it has been under the editorship of Dr. Rignano alone.

During these two decades it has appeared uninterruptedly, even throughout the period of war, when the publication of so many journals was temporarily or permanently discontinued. Indeed, it was in the years of the great conflict that some of its most scholarly and forceful articles were written, many of them dealing with the causes of the war and with its political and economical consequences. To the impartiality of the editor and his advisers was due a notable series of such articles, prepared during the period of Italy's neutrality, written by scholars in England, France, Germany, Russia, Austria, Hungary, the United States, and the Scandinavian countries, and setting forth with great ability the views of the leaders in thought in these various parts of the world. Such material, important as it was then, will have even greater historical value in the years to come.

It would be presumptuous for this writer to attempt to compile from the impressive roll of contributors a brief list that should best represent the work of *Scientia* during these twenty years. Indeed, it would be difficult to find a man whose grasp of science, of history, of the arts, and of letters qualifies him to do so. There come to the mind at once, however, a few names that may serve to show the general standing of those who have made the journal what it is,—a compendium of the scientific thought of the time. These may be classified roughly as follows:

History of the Sciences: Bortolotti, Cajori, Carra de Vaux, Dreyer, Favaro, Heiberg, A. Loria, G. Loria, Karpinski, Kaye, Mieli, Milhaud, Vacca, and Zeuthen.

Philosophy of Science: Enriques, Poincaré, and B. Russell.

Astronomy: Abbott, Chamberlain, Crommelin, R. S. Dugan, Dyson, Eddington, Hagen, Jeans, Kapteyn, Lowell, MacMillan, Palatini, Puiseaux, Shapley, See, and Very.

Biology and Physiology: Bohn, C. M. Child, De Vries, Lillie, G. Lusk, Pearl, E. S. Russell, and J. A. Thomson.

Chemistry: Bruni, Ciamician, Crowther, Findlay, and Soddy.

Economics and Law: Barclay, Burdick, Cahns, T. N. Carver, J. B. Clark, I. Fisher, Gini, Knibbs, Mondiani, Parmalee, Price, Ryan, W. R. Scott, Seligman, and Virgilii.

Geology: G. H. Darwin, R. A. Daly, De Marchi, Gregory, C. R. Keyes, W. B. Wright.

Mathematics: Borel, Boutroux, Carslaw, Castelnuovo, Dickson, Carmichael, Jourdain, Marcolongo, Peano, Rey Pastor, and others included elsewhere in this list.

Physics: G. H. Bryan, De Broglie, Fabry, Larmor, La Rosa, E. P. Lewis, O. Lodge, Lorentz, Righi, Rutherford, Somigliana, and Zeeman.

Psychology: Claparede, C. Lloyd Morgan, Rignano, and Thorndike.

Sociology: Caetani, Cardinali, Guignebert, Kidd, Landry, Perozzi, and Pettazzoni.

To an American it is particularly interesting to see the number of names representing the western continents and to feel that the present century is giving to the New World a worthy place in the progress of science in its broadest aspect and in the history of its various branches.

Since *Scientia* is an international review of scientific synthetic thought, recognized throughout the world, our public and university libraries should have on their shelves the entire series, if complete sets are still available.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. A CHECK FORMULA FOR THE FIRST CASE OF OBLIQUE TRIANGLES.

By L. J. PARADISO, Lehigh University

In connection with the solution of the oblique triangle when two angles and a side are given, it is desirable to check the results by a convenient formula. The tangent law is frequently used to check. If A, B, c, are the given parts and an error is made in getting the logarithm of c in applying the law of sines, then although a and b are wrong in consequence, the law of tangents will not detect this error. Mollweide's formulas have also been used as a check in this case but they also fail to detect an error made in getting either the logarithm of c or the antilogarithm of a, or of b.

The formula for the tan $(A/2)$ in terms of the sides of the triangle is sometimes used as a check in lieu of one of those already mentioned. Although

this formula is free from the objections given to those above yet it is obvious that it is a tedious one to use merely for a check.

The following formula, sometimes given as an exercise, has been found to give satisfactory results when used as a check. It is not open to the above objections:

$$(a + b - c)/(a + b + c) = \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B .$$

It can be proved directly from the formulas for $\tan \frac{1}{2}A$ and $\tan \frac{1}{2}B$ in terms of the sides or from the law of sines as follows:

$$\frac{a + b - c}{a + b + c} = \frac{\sin A + \sin B - \sin C}{\sin A + \sin B + \sin C} = \frac{4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \cos \tfrac{1}{2}C}{4 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C} = \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B.$$

For we have

$$\begin{aligned}\sin A + \sin B - \sin C &= \sin A + \sin B - \sin(A + B) \\ &= 2\sin \tfrac{1}{2}(A+B)\cos \tfrac{1}{2}(A - B) - 2\sin \tfrac{1}{2}(A+B)\cos \tfrac{1}{2}(A+B) \\ &= 2 \sin \tfrac{1}{2}(A + B)[\cos \tfrac{1}{2}(A - B) - \cos \tfrac{1}{2}(A + B)] \\ &= 2 \cos \tfrac{1}{2}C(2 \sin \tfrac{1}{2} A \sin \tfrac{1}{2}B).\end{aligned}$$

Similarly for the denominator.

II. A Note on Partial Fractions

By Raymond Garver, University of Rochester

In any first or second course in the calculus, a topic of some importance is the reduction of a rational function of x, say $n(x)/d(x)$, to the sum of partial fractions whose denominators are factors of $d(x)$. If $d(x)$ can be factored completely into real, linear factors, the work can be carried through quite easily by any one of several familiar methods. But if there are quadratic factors of the form x^2+px+q $(p^2-4q<0)$, the work is much longer.

Thus to reduce the fraction

$$\frac{3x^3 - 4x^2 - 3x + 5}{(x^2 + x + 1)(x^2 - 2x + 3)} \quad \text{to the form} \quad \frac{Ax + B}{x^2 + x + 1} + \frac{Cx + D}{x^2 - 2x + 3},$$

we equate the two, since we have presumably proved that such a reduction is possible. Clearing of fractions, we have

$$(1)\quad 3x^3 - 4x^2 - 3x + 5 = (Ax + B)(x^2 - 2x + 3) + (Cx + D)(x^2 + x + 1),$$

which is an identity. There are then two usual ways of computing A, B, C, D. We may assign four convenient integral values to x, or we may equate corresponding coefficients. Either process leads to four equations in four unknowns, a fairly long problem in algebra.

It seems possible to simplify the work by giving x an imaginary value which makes one of the quadratic factors vanish. The work can be arranged so that we never have to find this root explicitly at all. Thus, if we put $x=x_1$, where x_1 is a root of $x^2+x+1=0$, we have $x_1^2=-x_1-1$, $x_1^3=-x_1^2-x_1=1$; and equation (1) reduces to

$$(2)\qquad x_1+12=(Ax_1+B)(-3x_1+2)=-3Ax_1^2+(2A-3B)x_1+2B,$$
$$=(5A-3B)x_1+(3A+2B).$$

Equating real and imaginary parts, $1=5A-3B$, and $12=3A+2B$, which gives $A=2$, $B=3$. In this example we can obtain $C=1$, $D=-4$, by inspection.

This method also has the advantage that it can be used to derive a formula for the reduction of any fraction similar to our example. Thus if we have

$$(3)\qquad \frac{ax^3+bx^2+cx+d}{(x^2+p_1x+q_1)(x^2+p_2x+q_2)}=\frac{Ax+B}{x^2+p_1x+q_1}+\frac{Cx+D}{x^2+p_2x+q_2}$$

where $p_i^2-4q_i<0$ and A, B, C, D are to be determined; and if we go through the above process we obtain

$$(4)\qquad A=\frac{d(p_1-p_2)+(aq_1-c)(q_1-q_2)+(ap_1-b)(p_1q_2-p_2q_1)}{(q_1-q_2)^2+(p_1-p_2)(p_1q_2-p_2q_1)},$$
$$B=\frac{[q_1(aq_1-c)+dp_1](p_1-p_2)-[q_1(ap_1-b)+d](q_1-q_2)}{(q_1-q_2)^2+(p_1-p_2)(p_1q_2-p_2q_1)},$$

where the denominator does not vanish if the factors in the denominator of (3) are distinct, as we have tacitly assumed. The substitution in the formulas is easier than at first appears, since many of the terms are repetitions. If necessary, C and D can be found by interchanging subscripts, though in our example this was hardly necessary.

III. Is There A Student Standard of Truth?

By E. V. Huntington, Harvard University

Professor Osgood, in his letter in this Monthly for April, 1927, page 205, states a position in regard to the "truth" of a mathematical theorem which, as far as I know, is entirely new in the history of this Association. His position is that there is one standard of truth for a student and another standard of truth for a master of the science.

The occasion for this novel statement was the recent review by H. J. Ettlinger of F. S. Woods' *Advanced Calculus* (this Monthly, January, 1927 pp. 40–43). In this review, Professor Ettlinger had protested, a bit strenuously, against a certain theorem, which, according to hitherto accepted standards,

is *false*. Professor Osgood defends the theorem in question, saying: "The trouble with Ettlinger's review is that it fails to recognize *what, to the student, truth is*." In other words, what is true for a master is not necessarily true for a student, and what is false for a master is not necessarily false for a student.

I confess I am just old-fashioned enough to "view with alarm" any such tampering with two of the most sacred, well-established, terms in our mathematical vocabulary. What is any mathematical theorem if it is not something which is either definitely true or definitely false? If the truth of a theorem is to be dependent on whether the reader is a "student" or not, who is to decide where the line between student and master is to be drawn? If there is to be no objective mathematical standard of truth, what authority is competent to discriminate between the type of reader for which a given theorem is "true" and the type of reader for which it is "false"?

In particular, what is to be the position of a college text-book in regard to truth? A college text-book is written not only for students but also for teachers, and to a certain extent for the public. Is it to contain only a special brand of truth which is supposed to be suitable for a "student" or is it to present the real truth, in the sense in which mathematicians, and the general public, have hitherto understood that term? I mention the general public, because it seems to me important, in these days, not to risk doing anything which might disturb the confidence of the public in the truth of mathematical theorems.

It must be clearly understood, of course, that this question concerning the use of the terms true and false has nothing to do with the pedagogical problem of the selection of proper material for student use. A theorem may be true, and yet be "over the heads" of students at a certain stage. The point I wish to make is that a false theorem (unless explicitly studied as such) is of doubtful value to a student at any stage. Over against Professor Osgood's personal testimony on this point, we have the (probably more typical, student experience narrated by Benjamin Graham in this MONTHLY for June 1917, pp. 265–271. For myself, I agree heartily with Professor Carver's Editorial Note in the MONTHLY for April, 1927, p. 206; it surely is not too much to hope that we can "give the reader only so much of the truth as he is likely to understand, without making any statement which is false even from the most rigorous point of view."

Further references may be found in my paper *On setting up a definite integral without the use of Duhamel's theorem* (This MONTHLY, June, 1917, pp. 271–275, where, on page 273, it should have been noted that "the required quantity p" is supposed to be positive), and in Professor Ettlinger's paper on *A simple form of Duhamel's theorem* (This MONTHLY, vol. 29, 1922, pp. 239–249). These two papers, I may say, are typical of two quite opposite tendencies in the discussion of this highly inflammable subject. Discussion is the life of science, and I trust there will be more of it before the fire is finally quenched.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS

Advanced Calculus. By W. F. OSGOOD. New York, The Macmillan Co., 1925. 16+525+4 pages. Price $6.00.

When an American teacher plans to give a second year of work in calculus, he is likely to do one of three things: first, he may pick over the forgotten morsels of the books used in the first course in calculus or a similar book, such topics for instance as evolutes, convergence of series, application of the calculus to the determination of asymptotes for curves given in polar coordinates, approximate integration or such topics always found in any self-respecting text but regularly slighted in the first course even by the author; or secondly, he may give an eclectic course without text or printed exercises, covering various topics in the theory of real variables, or special solutions of differential equations, or other topics, not adequately treated in any elementary calculus text; or thirdly, he may select the best text on "Advanced Calculus" that he can find, and except for the conscious omissions of certain chapters follow this text with fidelity. In this third case, unless he adopts the familiar "Goursat-Hedrick," he is almost sure to turn to Cambridge, Massachusetts, and select either a green or red or blue text labelled simply "Advanced Calculus."

Supposing that the teacher makes his selection on something other than his taste in colors, he may read the prefaces in turn of Wilson's, Osgood's and Woods' books. No one of these authors suggests the possibility of other texts of this title existing, in this country or abroad, so that differential claims are not made. Osgood's preface opens with the assertion that "Any course in advanced calculus must deal with partial differentiation and multiple integrals, with systematic integration, with improper integrals, and to some extent with complex quantities;" but this remark doubtless sounds more dogmatic than intended by the distinguished author; for certainly several other reputable mathematicians do not act on this basis.

Professor Osgood is to be commended for this scholarly work covering an imposing array of important topics all treated with a clarity of style and economy of language that, in the opinion of the reviewer, set this book in these respects above any others in the American field. The skill in writing equations in just the form most readily applied, to mention but one feature, while not of a sort to catch the eye, shows not merely familiarity with the subject but also an interest in and attention to the psychological processes of the student approaching the subject for the first time, only too rare in the more serious American texts. A concern for the logical niceties of the subject is what the

reader is prepared to find in all of Osgood's work, and he is not here disappointed. What is likely to surprise the teacher who examines the book is that so many topics, usually treated carelessly or reserved for the rigor of a graduate course, are neither avoided nor rendered formidable. The book is not a review from a higher view-point of elementary calculus, most of the material here given being entirely new to the student for whom this text is intended. The problems are numerous and not excessively difficult, the print is attractive and misprints are rare although, for example, Rodrigues is referred to in the text and index as Rodigues.

Among the topics deserving of special interest on account of their relative novelty in a work of this sort, may be mentioned Mercator's chart, Lagrange's multipliers, Landen's transformation of the elliptic integral, the flow of electricity in conductors, irrotational flow of an incompressible fluid, the equation of continuity (in connection with flux across a surface), Stokes's theorem in vector form, small oscillations of a system with n degrees of freedom, zonal harmonics, the differential equation of the vibrating membrane, Hamilton's principle, entropy, the gamma function, conformal mapping.

Reviewers have been in the past severely criticised by more than one irate author who insisted that the reviewer's own opinion as to desirable subject matter in a text is wholly beside the point, and that only the degree to which an author achieves his own purpose is to be considered. The present review, from this stage on, will consider the text under discussion from what is intended to be the college teacher's point of view. Admitting a fine piece of logic and exposition, the reviewer cannot forget this is a college text book, intended for use in classes, not a treatise nor a disquisition for the specialist who may have like interest with the author.

Under the probability of being misunderstood, the reviewer feels compelled to state that this excellent text gives the impression of being fitted for a rather uncommon group of students, uncommon as to preparation and outlook. The students are assumed to have previous knowledge and adequate understanding of the theory of linear dependence for which reference is made to Bôcher's *Algebra*. Even more definitely is assumed a fair mastery of all convenient topics in Osgood and Graustein's *Analytic Geometry* and in Osgood's *Introduction to the Calculus*. For instance, the student is supposed to know (of course references are given to these previous texts) the parametric equation for a system of confocal ellipses and hyperbolas, the law of the mean in integral calculus, the theory of simple harmonic motion, and Duhamel's theorem. On the other hand, it is regarded in this text as necessary to begin with the first definitions of polynomials, the degree of a polynomial, etc., and close this rather extensive work with a short chapter on complex numbers, assuming complete ignorance of the subject on the part of the student. Furthermore the

reader is within the covers of this book expected first to meet Taylor's theorem, L'Hospital's rule, double integrals, reduction formulas etc., so far as it appears from internal evidence. For a student brought up on Granville or an equivalent text, the coordination is not ideal. However this is a matter easily capable of adjustment.

A more serious disadvantage, from the point of view of the smaller college which cannot hope to compete with the many special attractions of Harvard, is the following. A second course in calculus is usually in practice designed either as a required course for engineers, physicists and chemists, or as an elective for students planning to continue toward university work. The small college seldom has a large group of students electing one more year of analytic equipment for general application to geometry or mathematical physics. For engineers of the usual sort, this text shows a brave and interesting effort, but one which the reviewer regards as at present futile. There is too much emphasis on logical structure and on general principles, and the problems are rather too difficult to prescribe as a final mathematical dose to engineering students carrying fifteen hours a week of laboratory. On the other hand for the future mathematician eager to prepare himself at once for special work, the scope of the subjects here covered is discouraging. Many little snatches of large theories are suggested but the work appears choppy, particularly so in contrast to Wilson's book. Many of the topics are illustrated by very few exercises, sometimes by none at all. The epsilon-delta notation that every good mathematician probably enjoys, and which is so beautifully treated in an "elementary" manner in Hardy's *Pure Mathematics*, is consciously avoided, just as one tries to avoid mathematical induction with a poor freshman class in college algebra. This book does not of course compete with Edwards' *Integral Calculus* in exhaustiveness in the traditional subjects of integral calculus, or with the better French "Cours d'Analyse" in developing mathematical theory. It does not attempt either.

If Arts students at Harvard can and do complete most of the substance of this book early in their undergraduate career, and can follow up their undergraduate work by extensive courses in projective geometry, linear transformations, differential equations, analytic mechanics, potential theory and function theory, they are to be envied; but if after all this they cannot define continuity in sharp cut epsilon-delta notation, they will still be pitied by many persons. If engineering students at Harvard appreciate the many logical features of this book, then the other engineering schools of the country are missing something very fine.

ALBERT A. BENNETT.

New Methods in Exterior Ballistics. By F. R. MOULTON. Chicago, University of Chicago Press, 1926. vi+257 pages. Price $4.00.

A projectile in its flight through the resisting medium, air, traces a path which cannot be algebraically expressed in terms of any known functions. Yet that path must be subjected to fairly exact mathematical treatment, if our big guns are to be accurately aimed and laid.

The fundamental differential equations of motion of a projectile in flight have been known ever since the days of Euler. Up to the World War, the progress of ballistics had consisted in devising approximate algebraic expressions whereby to wrench these equations into soluble form. These approximations have been improved and improved, but still could not keep pace with the development of modern artillery. So that finally Prof. F. R. Moulton of the University of Chicago, while serving as a Major in the U. S. Army during the World War, cut the Gordian knot by going back to the Eulerian equations and solving them in their original and exact form by means of numerical integration. Thus he laid the conerstone for an entirely new science of ballistics, and accordingly no one is better qualified than he to recount the pioneer steps of this new science. As such an account, his book is an invaluable addition to the history of mathematical physics.

Morse and Bell laid such a complete groundwork in their invention and development of the telegraph and telephone respectively, that these inventions have been able to progress beyond all resemblance to the original device. And this is equally true of Moulton and Ballistics, and is equally to his credit. His book recounts little beyond his own work of 1918, yet this work was so new and revolutionary at that time, that he may be pardoned the poetic license involved in entitling his book, published in September, 1926, *New Methods in Exterior Ballistics*. Col. Tschappat's treatise on ballastics in vol. 30 of the Encyclopedia Brittanica, published in 1922, and even the reviewer's own book, published in 1921, are more up-to-date than Moulton's *New Methods*, published in September, 1926.

The seven years since Major Moulton left the service have been crammed with progress. An orthogonal curvilinear coordinate system was officially substituted for Siacci's tangent plane (p. 10) in 1920, and consequently the curvature of the earth (p. 115) is no longer corrected for. The differential corrections, to which Professor Moulton devotes nearly the whole of Chapter IV, have been obsolete since 1919; being superseded by the Bliss system, in which Moulton devotes but one section (p. 134). Furthermore, in the winter of 1919–1920, all the Bliss symbols were changed, Franklin devised for the Bliss formulas a derivation much more interpretable physically than Bliss' adjoint system, and Gronwall discovered a new first integral which revolutionized the computations. Moulton says (p. 141) that the Bliss method "is not adapted

to determining all the terminal variations of a trajectory, or of determining even one variation for a series of points along a trajectory;" yet the Franklin physical derivation makes clear that the Bliss method *is* adapted to determining all the terminal variations, and its use for determining any variation for a series of points (anti-aircraft) was demonstrated to be quite feasible by Hedrick in 1922. Of all this basic development, Professor Moulton makes no mention.

Even such slight matters as his measuring azimuth from the south (p. 15) instead of north, his extending the z-axis to the right (p. 8) instead of to the left of the xy plane, and his use of F (p. 9) instead of E for the resistance function, put upon a reader trained under present official conventions an unnecessary burden of constant translation of signs and symbols.

The reviewer cannot speak with authority on the chapter on "Motion of a Rotating Projectile" (p. 172), but it does not appear to bear much resemblance to, or take into account, the very complete theory built up by Capt. R. H. Kent, as a result of painstaking research at Aberdeen from 1919 to date.

The reviewer has no intent to disparage Professor Moulton's colossal recasting of an entire science, nor the value of his book as an historical repository; but he feels that a more appropriate title would be "Wartime Contributions to Ballistics," and that, if the author had used more modern symbols throughout his book, this storehouse of information would be more available to those whose experience with the subject is post-armistice.

ROGER SHERMAN HOAR.

Mathematics of Accounting. By A. B. CURTIS and J. H. COOPER. New York, Prentice-Hall, Inc., 1926. xii+397 pages. Price $5.00.

This book is intended to develop an accountant from a beginning of only arithmetic up to a point where he can decide the purchase price of a bond to yield a given investment rate. All this is done by the use of set forms under which each arithmetic computation can be made. Teachers of the mathematics of finance in college will find a few suggestive problems.

C. F. CRAIG.

College Geometry. A second course in plane geometry for colleges and normal schools. By NATHAN ALTSHILLER-COURT. Richmond, Va., Johnson Publishing Company, 1925. xiii+254 pages.

This book is based on a series of lectures given by the author at the University of Oklahoma. It has for its object the presentation of material for an advanced course in plane geometry, comparable in extent with the traditional college course in advanced algebra. In view of the well known general lack of adequate preparation among our high school teachers of geometry, a course

which would give greater facility in the solution of so-called "originals and constructions" should be a welcome addition to our college curriculum.

While the present work is in no sense a book of methods, it is, after all, a sort of trail blazer, and the author has set about his presentation of such theorems and problems as he has chosen to explain, in a way to make of them patterns to be followed in the solution of similar or related exercises. Indeed in such solutions he is most detailed in his proofs, leaving little for the student to supply beyond the explicit wording of extremely elementary theorems of plane geometry. This very elaborateness of solution might be counted a serious fault in a text for a college student, save for the redeeming feature that over 500 exercises are given for the development of the student's own ingenuity.

The first chapter is concerned with the method of attack in the solution of geometric problems, special attention being given to geometric loci and problems involving indirect elements, for example, the difference of two sides of a triangle as a given quantity. The use of similar and similarly placed figures with their properties is also stressed.

The next three chapters are devoted to "Properties of a Triangle." To the immature student, whose high school geometry leads him to think he knows all there is to know about the triangle, these chapters will bring a revelation, and may cause him to exclaim with Crelle, "It is indeed wonderful that so simple a figure as the triangle is so inexhaustible in properties. How many as yet unknown properties of other figures may there not be." Certainly such words as nine-point circle, Euler line, cyclic quadrilateral, sound strangely in the ears of the pupil from our ordinary high school classes, while Ptolemy's, Stewart's, and Feuerbach's theorems may show him that the science of geometry is the product of the minds of many men.

After a brief chapter on "Harmonic Section," there is introduced a discussion of the properties of circles. Inverse point, poles and polars, radical axes, coaxial circles and centers of similitude all receive attention. A chapter on "Inversion" and one on "Recent Geometry of the Triangle" complete the book.

The author's style is simple and direct, his notation is explicit and consistent. His choice of material is interesting and the exercises are varied enough to intrigue the student into further study. Typographically the book is excellent and the index seems adequate.

Let us revert now to the "Introduction," which, it might be said in passing, is not an introduction, but rather a second preface, in no sense the beginning of the subject matter. The author's expressed hope that "the vast army of high school teachers · · · might be worthily represented in this field of research," would seem to have greater chance of fulfillment if the student could learn a few

historical facts, even in footnotes. A knowledge of the chronological development of the subject would surely be worth while to any investigator.

But if the absence of dates seems only an omission and the student may be expected to search elsewhere for the genesis of his theorems, yet the misleading statement is not so to be excused. The author says: "In the last quarter of the nineteenth century, modern geometry became enriched by a splendid addition known as the 'geometry of the triangle' ," · · · "Take, for instance, · · · Ceva's theorem, the nine-point circle, · · · ." This is the only mention of time in the entire work, and as the extract from the following paragraph shows, the author is by no means confining himself to the period he mentions. Furthermore the term "modern geometry" taken in connection with these sentences is quite sure to catch the student's attention, and, finding in juxtaposition in the text, Simson's line, Menelaus' theorem, and Ceva's theorem, he may naturally consider them developments of the same generation and ask his teacher if, "the gentlemen mentioned were by any chance classmates at college."

HELEN B. OWENS.

Advanced Algebra. By H. E. BUCHANAN AND L. C. EMMONS. Boston, Houghton Mifflin Co., 1925. viii+185 pages.

This charming little book is a veritable story of adventure. The elusive Unknown is pursued through thickets of polynomials and labyrinths of determinants until, stayed by the calming influence of logarithms, through arithmetical and geometrical progression it attains an annuity at the end.

The figure is sober and nearly accurate. The enchanting quality of the book is largely due to its arrangement; to the many things which are not found in it; and to the introduction of new material in those early chapters which are supposed to furnish a review of parts of the more elementary algebra. Among other things this new material includes the use of the derivative, the definition of the conics with their equations, and maxima and minima.

The chapter on polynomials contains what the student needs for the solution of numerical quantics; Descartes' rule and Newton's and Horner's methods. In chapter VI, mathematical induction is defined and illustrated before its use in proving the binomial theorem. Permutations and probability are treated with Spartan meagreness. All necessary theorems of determinants are given. The explanation of logarithms and of the method of using the four place table are as nearly fool-proof as can be imagined. The final chapter contains the progressions and applications to annuities, which students always find interesting, and either the first or the final proof that a table of logarithms may really be of value. There is an appendix of nine pages devoted to scales of notation.

It is conceivable that no teacher ever willingly admits that the textbook written by another hand is quite perfect. If that is true, while admitting all the merits of this book, he would probably lament that the treatment of permutations and of probability is almost sketchy and that nine pages are too few for these two subjects. He might also regret that more problems are not given in the chapter on determinants. He would surely find the introductory chapters so interesting that he would not willingly omit any part of them, and then he would be puzzled to cover what his conscience demands, in fifty hours, with his small army of conscripts who would gladly evade the carnage of this conflict.

However it seems impertinent and intrusive to attempt a review of this most excellent text, since the preface by the authors and the introduction of the editor say all that needs to be said, and with the clarity and precision that should distinguish the mental processes of a mathematician.

W. W. LANDIS.

Esercizi di Geometria Analitica raccolti a cura di AMEDEO AGOSTINI *ed* ENEA BORTOLOTTI. Parte prima, 1925, x+230 pages; Parte seconda, 1926, 469 pages. Bologna, Nicola Zanichelli editore.

These two volumes are companions to the *Lezioni di Geometria Analitica*, from the pen of Ettore Bortolotti, and reviewed in this *Monthly*, vol. 31 (1924) pp. 98, 99. As the title indicates, the *Esercizi* contain problems and exercises in the elementary and the higher field of analytic geometry. Many of them are new, others are classic problems of interest. To each problem is given the answer, accompanied when thought desirable by hints as to its solution. Here and there are given also theoretical matters not found in the *Lezioni*. The first volume deals mainly with points and right lines in the plane and in space; the second volume with conics, with surfaces and curves in space, and with the elements of differential geometry. The volumes will be found very serviceable to teachers and students in the fields of which they treat.

FLORIAN CAJORI.

The Theory of Measurements. By LUCIUS TUTTLE AND JOHN SATTERLY. London, Longmans, Green, and Co. 1925. xii+333 pages.

In the preface it is made quite clear that this book is intended, primarily, for students of physical science. Its chief use is as a supplement to the usual laboratory manual. The idea which serves to connect the subject matter into a consistent whole is *physical measurements*. The mathematics in it is brought in incidentally as it is needed. If it were judged only by the mathematics in it, it would appear as a collection of numerous disconnected subjects. This would be an unfair method of judging it.

There are twenty-two chapters followed by a set of miscellaneous exercises for student practice, a list of answers to these miscellaneous exercises and the numerous other exercises scattered throughout the book, nine short tables some of which are not ordinarily available for convenient reference, a short bibliography and finally an index.

The mental equipment of the student for whom this book is designed is not above that of a high school graduate. The mathematics presupposed is described as follows:

"A good course in algebra as far as the solution of equations of the first degree; also a sufficient knowledge of plane geometry to include the properties of perpendiculars, equal triangles, isosceles and similar triangles, the theorem of Pythagoras, and the properties of similar figures."

Among the chapter titles are the following: Weights and Measures (9 pages), Angles and Circular Functions (10 pages), Significant Figures (13 pages), The Slide Rule (10 pages), Graphic Representation (26 pages), Graphic Analysis (22 pages), Accuracy (11 pages), Measurements and Errors, Statistical Methods (20 pages), Indirect Measurements (11 pages), Least Squares (21 pages). Each is taken up in much detail. To one attempting to read the book through, independently of any simultaneous laboratory work, it gives much the same impression as one would get if he attempted to read through a dictionary; but that is not an adverse criticism any more than it is an adverse criticism of a dictionary.

The present reviewer has not had any recent experience with students in a physical laboratory, so that he does not feel competent to pass judgement from that point of view. His impression, however, is that it will be found to be very useful in laboratory courses, and it may be found advisable for the science departments of our colleges to require their beginning students to possess copies of it for convenient reference, and possibly a regular course of class instruction might be given to such students with this as a textbook. Evidently that is the practice of the author.

A somewhat hasty reading revealed no fundamental errors. The mathematics which the author has applied to the problems in hand is sound. A student of this book who should afterwards take up systematic courses in higher mathematics would not be forced to discard any concepts acquired from this book, but would find his acquisition of the advanced abstract mathematics helped by his experience with these concrete applications.

I believe the author and publishers have served a good purpose by giving this book to the educational world.

GEO. GAILEY CHAMBERS.

A Comment on the Review of Chambers "Statistical Analysis"

In the review of Dr. G. G. Chambers book "An Introduction to Statistical Analysis" in the January number of this Monthly, the reviewer has overlooked the fact that it was written to meet a very particular situation which exists at the University of Pennsylvania, as is stated in the preface. All members of the freshman class in the School of Education are required to take a half-year course in elementary statistics. The entrance requirements to this school are such that the student need only present two years of High School mathematics. Such a condition makes it necessary to have a text in which only very elementary mathematical concepts are used. The aim of the course is to give the students a sufficient background of the methods and terms in statistics so that they may be able to read intelligently articles and books dealing with educational research, and also to be able to make the necessary computations in a later course in "Tests and Measurements." In addition to this course there are a number of advanced courses in statistics given by the Mathematics department.

This text was written to meet the need of such a group of students and was not intended to be a text in mathematical statistics such as would be used with students who had had some college courses in mathematics and had thus acquired a mathematical background. The amount of space devoted in the first chapter to approximations might seem to be rather large, but to one who has taught the course it is not too great, as most of the students come with the idea that a measurement is an exact quantity.

It is the aim of the book throughout to impress the students with the fact that to go far in statistics a great amount of mathematics is required and that they are only getting an introduction so as to make them able to comprehend the meaning of the terms and methods of computation which will be used in later work.

H. M. LUFKIN.

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) the titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

American Journal of Mathematics, volume 49, no. 1, January 1927: "Stability and the equations of dynamics" by G. D. Birkhoff, 1–38; "Quaternary quadratic forms representing all integers" by L. E. Dickson, 39–56; "Reduction formulas for the number of representations of integers in certain quadratic forms" by E. T. Bell, 57–66; "Generalization of certain theorems of Bohl" by F. H. Murray, 67–86; "Applications of the determinant and permanent tensors to determinants of general class and allied tensor functions" by C. M. Cramlet, 87–96; "Transformations leaving invariant certain partial differential equations of physics" by R. D. Carmichael, 97–116; "Transformations leaving invariant the heat equation of physics" by J. B. Goff, 117–122; "The application of fractional operators to func-

tional equations" by H. T. Davis, 123–142; "Classification of quadrics in hyperbolic space" by J. Pierpont, 143–151.

Annali di Matematica, ser. 4, volume 4, November 1926–1927: "Transformations of relations between numerical functions" by E. T. Bell, 1–6.

Bulletin of the American Mathematical Society, volume 33, no. 1, January–February 1927: "On the metrization problem and related problems in the theory of abstract sets" by E. W. Chittenden, 13–34; "A theorem on factorization" by D. H. Lehmer, 35–38; "On a problem in closure" by Virgil Snyder, 39–43; "A theorem concerning direct products" by L. Wiesner, 44; "A cubic curve connected with two triangles" by H. Bateman, 45–50; "On the integration in finite terms of linear differential equations of the second order" by J. F. Ritt, 51–57; "On small deformations of curves" by C. E. Weatherburn, 58–62; "Integers represented by positive ternary quadratic forms" by L. E. Dickson, 63–70; "A diophantine automorphism" by E. T. Bell, 71–80; "The Cauchy-Heaviside expansion formula and the Boltzmann-Hopkinson principle of superposition" by F. D. Murnaghan, 81–89; "Singularities of the Hessian" by T. R. Hollcroft, 90–96; "Three theorems on closure of biorthogonal systems of functions" by R. E. Langer, 97–105; "A connected and connected in kleinem point set which contains no perfect subset" by B. Knaster and C. Kuratowski, 106–109.

Bulletin of the American Mathematical Society, volume 33, no. 2, March-April 1927: "A mathematical critique of some physical theories" by G. D. Birkhoff, 165–181; "An assemblage theoretic proof of the existence of transcendentally transcendental functions" by J. F. Ritt and E. Gourin, 182–184; "The most general closed point set over which continuous functions may be defined by certain properties" by G. T. Whyburn, 185–188; "Invariant relations" by F. H. Murray, 189–192; "Determination of the number of subgroups of an abelian group" by G. A. Miller, 193–194 "The asymptotic osculating quadrics of a curve on a surface," by E. P. Lane, 195–200; "A new characterization of plane continuous curves" by W. L. Ayres, 201–208; "A method for accelerating the convergence in the process of iteration" by C. C. Camp, 209–220; "A quadratic algebra and its application to a problem in diophantine analysis" by G. E. Wahlin, 221–231; "Functions expansible in series" by L. E. Ward, 232–234; "Invariants of a poristic system of triangles" by J. H. Weaver, 235–240.

The Monist, volume 37, no. 1, January 1927: "Mathematical Reality" by J. B. Shaw, 113–119; "Mathematics and natural science" by A. Dresden, 120–130; "Infinity and the infinitesimal" by W. Parkhurst and W. J. Kingsland Jr., 131–150.

Proceedings of the National Academy of Sciences, U. S. A., volume 13, no. 2, February 1927: "On the closure of certain assemblages of trigonometrical functions" by N. Wiener, 27–28; "On a type of Lorentz transformations" by G. Y. Rainich, 29–30; "Cyclicly connected continuous curves" by G. T. Whyburn, 31–37; "Displacements in a geometry of paths which carry paths into paths" by L. P. Eisenhart and M. L. Knebelman.

Proceedings of the National Academy of Sciences, U. S. A., volume 13, no. 3, March 1927: "Dynamical economics" by C. F. Roos, 145–150; "On the proof of Shepard's corrections" by E. B. Wilson, 151–155; "Note on a generalization of Taylor's series" by D. V. Widder, 156–159; "A theory of matter and electricity" by G. D. Birkhoff, 160–164; "The hydrogen atom and the Balmer formula" by G. D. Birkhoff, 165–169; "Groups generated by two operators of order three whose product is of order six" by G. A. Miller, 170–174.

Rendiconti del Circolo Mathematico di Palermo, volume 50, September–December 1926: "The general class number relations contained in Jacobi's theta formula" by E. T. Bell, 368–374.

Science, volume 65, no. 1676, February, 1927: "A mathematical critique of some physical theories" by G. D. Birkhoff, 147–149.

Transactions of the American Mathematical Society, volume 29, no. 1, January 1927: "Sets of independent postulates for the arithmetic mean, the geometric mean, the harmonic mean, and the root-mean-square" by E. V. Huntington, 1–22; "Irregular difference systems of order two and the re-

lated expansion problems" by M. H. Stone, 23–53; "Some problems in the theory of interpolation by Sturm-Liouville functions" by C. M. Jensen, 54–79; "Singular ruled surfaces in space of five dimensions" by E. B. Stouffer, 80–95; "On a type of completeness characterizing the general laws of separation of point-pairs" by C. H. Langford, 96–110; "Integers and basis of a number field" by N. R. Wilson, 111–126; "Implicit functions and their differentials in general analysis" by T. H. Hildebrandt and L. M. Graves, 127–153; "Applications of the theory of relative cyclic fields to both cases of Fermat's last theorem" by H. S. Vandiver, 154–162; "Riemann integration and Taylor's theorem in general analysis" by L. M. Graves, 163–177; "Alternatives to Zermelo's assumption" by A. Church, 178–208; "On a general theorem concerning the distribution of residues and non-residues of powers" by J. M. Vinogradov, 218–226; "On convergence factors in multiple series" by C. N. Moore, 227–238.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

THE MATHEMATICAL AND PHYSICAL SOCIETY OF THE UNIVERSITY OF TORONTO, Toronto, Canada.

The Mathematical and Physical Society was founded forty-five years ago and is almost half as old as the University of Toronto which this year is celebrating its hundredth anniversary. The fact that the Society has been decidedly successful in its functions of furthering interest in mathematics and physics and providing its members with certain social advantages is vouched for by its steadily increasing membership and popularity with graduates and under-graduates. The average attendance this year has been about one hundred and fifty.

The regular meetings are held fortnightly on Thursday afternoons in the Physics Building. Following refreshments it is the custom to have some outstanding man address the members on a subject of general interest. This year the executive has been very fortunate in securing for this purpose several leading members of the Faculty as well as a number of graduates and undergraduates who gave interesting talks. The very keen interest taken in the social activities of the society by the members has had the inevitable effect of making every event from the "Hike" to the "Open Meeting" an unqualified success.

Executive for 1926-1927

Hon. Pres. Dean A. T. DeLury, President C. A. Peachey 4th Yr., Vice-Pres. Miss M. Annetts 3rd Yr., Corr. Sec. Miss A. Keast 4th yr., Treasurer C. L. Bates 2nd Yr., Record. Sec. Miss W. Woollcombe, 3rd Yr., Grad. Rep., Miss M. Westman, B. A., 4th Yr. Rep., W. L. McCutcheon, 3rd Yr. Rep., R A.. Blythe, 2nd Yr. Rep., W. D. Douglas, 1st Yr. Rep., Miss I. Winter, N. M. Burns.

October 9, 1926. Annual hike to welcome freshmen and freshettes.

October 15. Opening meeting. "Outstanding scientific societies" by Professor C. McLennan, Director of the physics laboratory at Toronto.

October 29. "Newtonian philosophy" by Alfred Baker, Emeritus professor of mathematics.

November 13. Annual At-Home of the M. and P. Society.

November 26. A meeting devoted to impromptu talks by members of the society on subjects of scientific interest.

December 9. Voting on changes in the constitution. "Alloys and their properties" by Mr. Taylor. "Astronomical legends" by Miss Stone.

January 21, 1927. "Non-euclidean geometry" by Professor A. T. DeLury, Honorary President and Dean of the Faculty of Arts.

February 5. Annual skating party.

February 25. Open meeting. The relativity drama which appeared in this Monthly for June 1926 was presented with a few local insertions. There were also several musical numbers and readings. Refreshments and dancing.

March 17. Bicentenary of the death of Sir Isaac Newton. Addresses:—"Newton's optical view" by Professor J. C. McLennan of the department of physics. "Mathematical work of Newton" by Professor J. Chapelon of the department of mathematics. "Contributions of Newton to astronomy" by Professor Chant of the department of astronomy. This last meeting was open to the public and several hundred people attended.

(Report by Annie M. Keast)

The Mathematics Club of the University of Oklahoma, Norman, Oklahoma.

During the year 1925–1926, the officers of the Mathematics Club of the University of Oklahoma were Hazel Jones, president; C. E. Springer, vice president; Sarabeth Barbour, secretary-treasurer. The meetings were held in the afternoon and tea was served before the formal program began.

The following topics were discussed:

October 29, 1925. "Newton's methods of graphing curves" by Dr. Elsie J. McFarland.

November 12. "Complex numbers" by Kathleen Butt.

December 10. "Tetracyclical coordinates" by John C. Buxey.

February 4, 1926. "Bertrand Russell" by Mr. Howard Eaton, Department of Philosophy.

March 25. "History of the calculus" by Travis Groves.

May 6. "The geometrical properties of the hyperbola" by Wilma Gorton "Indeterminate equations" by Hazel James.

(Report by L. D. Montgomery, secretary)

Mu Theta Epsilon, University of California, Berkeley, California.

The following is the program of Mu Theta Epsilon, women's mathematical honor society of the University of California, Berkeley, California, for the year 1925–1926.

September 10, 1925. Mathematical recreations. Discussion.

September 23. "The logical foundations of algebra" by Mamie Giacomini and Lois Matzen. Welcome to new members. "The early history of Mu Theta Epsilon" by Dr. Pauline Sperry.

October 14. "Fundamental concepts of calculus" from J. W. A. Young's monographs. Discussion led by Bertha Vranna.

November 17. "Fundamental concepts of algebra" from J. W A. Young's monographs. Discussion led by Jessie Ramelli.

January 13, 1926. "Space" by Harriet Bowker. "The application of mathematics to statistics" by Elizabeth Lange.

February 10. "The determination of an orbit" by Dr. Sophia Levy.

March 10. "Functions of a complex variable" by Louise Kemp. "Roman methods of computation" by Beryl Britton.

March 24. Initiation of new members and banquet.

April 14. Summary of reconstructed history of Mu Theta Epsilon by Beatrice Ude. "The method of least squares" by Elizabeth Burroughs.

(Report by Harriet H. Bowker)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are expecially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3264. Proposed by L. S. Shively, Mount Morris College.

A square is subdivided into n^2 equal squares by drawing lines parallel to its sides, and all the diagonals of the squares thus formed are drawn. How many triangles are there in the configuration?

3265. Proposed by Norman Anning, University of Michigan.

Prove, or improve, the scout's rule for reading bicycle spoor: an inflexional tangent to the path of the hind wheel is an ordinary tangent to the path of the front wheel.

3266. Proposed by W. A. Jenkins, Chicago, Ill.

Cards are exposed one at a time, without being replaced, from an ordinary deck of playing cards. The draw ceases as soon as four cards of any one number (or Jack, Queen, or King) are exposed. Find the probability that the ith card exposed concludes the draw.

3267. Proposed by H. A. DoBell, Colgate University.

In a given triangle the line joining the ninepoint center and the orthocenter of the pedal triangle is parallel to the Euler line of the tangential triangle.

3268. Proposed by Tobias Dantzig, University of Maryland.

A and B are two fixed points on a conic and M and N, two variable points on the same conic, such that the cross-ratio $(A, B; M, N)$ with regard to the conic remains constant. Find the envelope of the line MN.

3269. Proposed by J. Rosenbaum, Milford, Conn.

On the sides A_1B_1, B_1C_1, C_1A_1, of a triangle, $A_1B_1C_1$, the points A_2, B_2, C_2 are taken so that

$$A_1A_2 : A_2B_1 = B_1B_2 : B_2C_1 = C_1C_2 : C_2A_1 = \lambda,$$

thus forming a second triangle $A_2B_2C_2$. In this manner, by repeating the operation on the successive triangles a sequence of triangles $T_1, T_2, \cdots$, is obtained. Find under what conditions will (a) T_n be similar to T_1; (b) the shape of T_n approach a limit as n becomes infinite. (c) Find the point to which the three vertices, A_n, B_n, C_n converge when λ is a positive, proper fraction.

3270. Proposed by Nathan Altshiller-Court, University of Oklahoma.

If a triangle is self-polar with respect to a circle, the center of the circle is the orthocenter of the triangle.

State this proposition in a projective form and give a direct proof of the proposition so stated.

3271. Proposed by H. L. Olson, Michigan State College.

Given a positive integer r; find all the positive integers of which r is a primitive root.

SOLUTIONS

3198[3186; 1926, 279]. Proposed by A. A. Bennett, Lehigh University.

Let the twelve letters, $a, b, c, \cdots k, l$, denote in some unknown order, the twelve distinct residue classes taken modulo 12. Assume that the complete multiplication of 144 products, giving each product

as a letter of the set, is known, but nothing further is given concerning the sum or difference of two letters. Show that each residue class is completely identifiable. Determine whether or not a like theorem holds for the case when the modulus is 16, 18, or 24.

SOLUTION BY H. D. GROSSMAN, Brooklyn, N. Y.

We are given a table of congruences which we shall call

TABLE I

$a^2 \equiv$	·	·	·	$ba \equiv$	·	·	·	·	·	·	·	·	$la \equiv$	·	·	·
$ab \equiv$	·	·	·	$b^2 \equiv$	·	·	·	·	·	·	·	·	$lb \equiv$	·	·	·
$ac \equiv$	·	·	·	$bc \equiv$	·	·	·	·	·	·	·	·	$lc \equiv$	·	·	·
·	·	·		·	·	·		·	·	·			·	·	·	
·	·	·		·	·	·		·	·	·			·	·	·	
$al \equiv$	·	·	·	$bl \equiv$	·	·	·	·	·	·	·	·	$l^2 \equiv$	·	·	·

where the second term of each congruence is one of the letters, $a, b, \cdots, l$, taken positively, and where (as hereafter in this solution) "modulo 12" is understood after each congruence.

Now consider another table,

TABLE 2

$0 \cdot 0$	$\equiv 0$		$1 \cdot 0$	$\equiv 0$	·	·	·	·	·		$11 \cdot 0$	$\equiv 0$		
$0 \cdot 1$	$\equiv 0$		$1 \cdot 1$	$\equiv 1$	·	·	·	·	·		$11 \cdot 1$	$\equiv 11$		
$0 \cdot 2$	$\equiv 0$		$1 \cdot 2$	$\equiv 2$	·	·	·	·	·		$11 \cdot 2$	$\equiv 10$		
·	·	·	·	·	·		·	·	·		·	·	·	
·	·	·	·	·	·		·	·	·		·	·	·	
$0 \cdot 11$	$\equiv 0$		$1 \cdot 11$	$\equiv 11$	·	·	·				$11 \cdot 11$	$\equiv 1$		

If we compare these tables, it is obvious that in Table 1, there is one and only one column in which the second terms are all alike and all equal to the common multiplicand, which we may assume to be a. Then $a=0$. Substitute this value for a wherever found in the table.

It is equally obvious that in Table 1, there is one and only one column in which each second term is equal to the multiplier of the first term. Let the common multiplicand in this column be b. Then $b=1$. Substitute this value for b wherever found in the table.

Consider Table 2 again or rather those parts of it appended:

$6 \cdot 0 \equiv 0$	$4 \cdot 0 \equiv 0$	$8 \cdot 0 \equiv 0$	$3 \cdot 0 \equiv 0$	$9 \cdot 0 \equiv 0$	$2 \cdot 0 \equiv 0$	$10 \cdot 0 \equiv 0$
$6 \cdot 2 \equiv 0$	$4 \cdot 3 \equiv 0$	$8 \cdot 3 \equiv 0$	$3 \cdot 4 \equiv 0$	$9 \cdot 4 \equiv 0$	$2 \cdot 6 \equiv 0$	$10 \cdot 6 \equiv 0$
$6 \cdot 4 \equiv 0$	$4 \cdot 6 \equiv 0$	$8 \cdot 6 \equiv 0$	$3 \cdot 8 \equiv 0$	$9 \cdot 8 \equiv 0$		
$6 \cdot 6 \equiv 0$	$4 \cdot 9 \equiv 0$	$8 \cdot 9 \equiv 0$			$2 \cdot 4 \equiv 0$	$10 \cdot 4 \equiv 4$
$6 \cdot 8 \equiv 0$			$3^2 \equiv 9$	$9^2 \equiv 9$		
$6 \cdot 10 \equiv 0$	$4^2 \equiv 4$	$8^2 \equiv 4$				

We already have all our zeros in Table 1. There will be one and only one column in which there are 6 zeros among the second terms. Let the common multiplicand in this column be c. Then $c=6$.

There are two and only two columns each having 4 zeros among its second terms. Let their multiplicands be d and e. The squares of these letters are both congruent to one of them. Let that one be d. Then $d=4$, $e=8$. Substitute these values for d and e wherever found in the table.

Proceeding similarly we will identify two other letters, say f and g, as 3 and 9 respectively.

Proceeding similarly we will find a pair of letters, say h and i, which will be equal in unknown order to 2 and 10. Their squares will be indistinguishable but the product of one of them by 4 is congruent to 8, that of the other is congruent to 4. The first is 2, the second 10.

The three letters remaining may be easily identified by checking their functions against these congruences:

$$5 \cdot 2 \equiv 10;\ 7 \cdot 2 \equiv 2;\ 11 \cdot 2 \equiv 10;\ 5 \cdot 3 \equiv 3;\ 11 \cdot 3 \equiv 9.$$

To prove the second part of the theorem, namely, that it is not true for any multiple of 8 or 9, it is necessary only to prove that even if all the other letters are identified, there will still remain a pair, say x and y, the symmetry of whose like functions will render them indistinguishable. That is, if

(1) when $f(x) \equiv x$, $f(y) \equiv y$, and

(2) when $f(x) \equiv y$, $f(y) \equiv x$, and

(3) when $f(x) \equiv z$, $f(y) \equiv z$, where $z \neq x$ and $z \neq y$,

it is clear that x and y are indistinguishable, therefore individually unidentifiable.

I shall endeavor to show that for any number of the form $8m$, $2m$ and $6m$ constitute such a pair of numbers, and that for any number of the form $9m$, $3m$ and $6m$ constitute such a pair.

Let $x=2m$, $y=6m$, and the modulus be $8m$.

$f(x)$ and $f(y)$ must be of the forms nx, ny (where $n \neq x$ and $n \neq y$), x^2, y^2, or xy. Now $n \equiv 0$, 1, 2 or 3 (mod 4).

If $n \equiv 0 \pmod 4$, then $2mn \equiv 0 \pmod{8m}$ and $6mn \equiv 0 \pmod{8m}$, satisfying (3).
If $n \equiv 1 \pmod 4$, then $2mn \equiv 2m \pmod{8m}$, $6mn \equiv 6m \pmod{8m}$, satisfying (1).
If $n \equiv 2 \pmod 4$, then $2mn \equiv 4m \pmod{8m}$, $6mn \equiv 4m \pmod{8m}$, satisfying (3).
If $n \equiv 3 \pmod 4$, then $2mn \equiv 6m \pmod{8m}$, $6mn \equiv 2m \pmod{8m}$, satisfying (2).

Further, $(2m)(6m) \equiv (2m)^2 \pmod{8m}$; and since $2m \equiv 1$ or $3 \pmod 4$ and $(2m)^2 \not\equiv 2m$ or $6m \pmod{8m}$, it follows that $(2m)(6m) \not\equiv 2m$ or $6m \pmod{8m}$.

Also $(2m)^2 \equiv (6m)^2 \pmod{8m}$, satisfying (3).

A similar proof may be worked out for moduli of the form $9m$.

Also solved by MICHAEL GOLDBERG.

3199[3187; 1926, 338]. Proposed by Frank Morley, Johns Hopkins University.

Taking six points, 1, 2, 3, 4, 5, 6, on a conic, let the areas of the triangles 612, 123, $\cdots$ be denoted by $a_1, a_2, \cdots, a_6$. Prove that the area, A, of the hexagon, formed by the points as ordered, is given by

$$(A-a_1-a_3-a_5)a_2a_4a_6 = (A-a_2-a_4-a_6)a_1a_3a_5.$$

SOLUTION BY OTTO DUNKEL, Washington University.

The area of the triangle formed by the points P_i, P_j, P_k is

$$\pm (i,j,k) = \pm \tfrac{1}{2} \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix}, \tag{1}$$

where the $+$ or $-$ sign is used according as P_i, P_j, P_k occur in positive or negative rotation. If (x_s, y_s) denotes a variable point, then $(z, j, k) = 0$ is the equation of the straight line through P_j and P_k. The equation of a conic passing through P_1, P_3, P_4, P_6 may be written

$$(1\ z\ 6)(z\ 3\ 4) - k(1\ z\ 3)(z\ 4\ 6) = 0, \tag{2}$$

and, if we set $z=2$ or 5, k is uniquely determined since all six points lie on the same conic (considered to be non-degenerate). Eliminating k from the two resulting equations, we have

$$(126)(234)(456)(135) - (231)(345)(561)(246) = 0. \tag{3}$$

It is geometrically evident that

$$A = (135) + (231) + (345) + (561) = (246) + (126) + (234) + (456). \tag{4}$$

If we eliminate (135) and (246) from (3) and (4), we have the equality in the problem.

Also solved by T. W. EDMONSON and HARRY LANGMAN.

3201[3189; 1926, 338]. Proposed by C. G. Latimer, Tulane University.

Find integers, h, k_i, c_i, n, where $|k_i| < h$, such that

$$\log(x+h) = \sum_{i=1}^{n} c_i \cdot \log(x+k_i) + \sum_{i=1}^{\infty} a_i z^i$$

where the a_i are constants and $z = b/f(x)$ where b is a constant and $f(x)$ is a polynomial of degree $\geqq 6$.

Borda's and Haro's series are similar to this except that they involve polynomials of the third and fourth degrees respectively.[1]

Remarks by E. B. Escott, Oak Park, Illinois.

This question is answered by me in a paper on *Logarithmic Series* in the *Quarterly Journal of Mathematics* (1910), pages 141–156.

In this paper (page 152) is the solution

$$\log(x+51) = \log(x+50) + \log(x+38) - \log(x+33) - \log(x+24) + \log(x+13) + \log(x+7) - \log(x-7) - \log(x-13) + \log(x-24) + \log(x-33) - \log(x-38) - \log(x-50) + \log(x-51) + 2[6983776800/(x^7 - 4214x^5 + 4716649x^3 - 1218798036x) + \cdots].$$

[Note: In the paper there is a misprint in the known term of equation (52). There should be two zeros annexed to the known term as given.]

Other equations of the 7th degree similar to (52) are those having the following roots: 6, 19, −30, −40, 49, 51, −55; 8, 47, −66, −75, 87, 133, −134; 13, 23, −38, −105, 110, 120, −123; and others may easily be obtained.

In the above paper are given several series where $f(x)$ is of degree 6. Many of these series may be transformed into series involving antitangents instead of logarithms. On this subject see L. E. Dickson's *History of the Theory of Numbers*, vol. 2, page 716.

These series may also be used for interpolation. The connection will be seen by taking the formula $\Delta^3 u_0 = u_3 - 3u_2 + 3u_1 - u_0$ and applying it to the function $u_n = \log(x+n+1)$.

3202[3190; 1926, 338]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Two given lines x, y are met in the points P, Q by a circle passing through their common point and through another fixed point in the plane. Find the locus of the point of intersection of the lines projecting P, Q upon y, x, respectively.

Solution by Otto Dunkel, Washington University.

Let A be the fixed point; then, since $\angle PAQ$ is equal to the angle between x and y (assumed to be different from 90°), the range of points P on x is projective with the range of points Q on y, and to the point at infinity on x corresponds the point at infinity on y. Hence the two sets of projecting lines, one set through the points P and the other through Q, are projective, and they have a self-corresponding ray in the line at infinity. Hence the locus of the points of intersection of corresponding rays is a straight line.

Also solved by Theodore Bennett, Michael Goldberg, and the Proposer.

3205[3193; 1926, 338]. Proposed by W. H. Rasche, Virginia Polytechnic Institute.

Prove that two coplanar, copolar triangles are homological; and, conversely, two coplanar, homological triangles are copolar.

Note.

(1) In the proof of the foregoing theorem, use only Euclidean geometry and the positive and negative signs to distinguish the internal division of a linear segment from its external division.

Note by the Editors

The theorem above is commonly called Desargues' theorem, and proofs of the theorem may be found in most texts on projective geometry. The usual proof is given by constructing a triangle outside of the given plane which is in perspective with the two coplanar triangles by two centers on a line concurrent with the three lines through corresponding pairs of vertices (see Cremona's Proj. Geom. p. 7; Mathews' Proj. Geom. p. 26).

---

[1] See Goursat-Hedrick *Mathematical Analyses*, vol. 1, p. 133.

A proof of the kind desired in the problem is as follows: Let AA', BB', CC' meet in O, and let BC and $B'C'$ meet in X and cut OAA' in D and D', respectively. Then $(D, B, C, X) \overline{\wedge} (D', B', C', X)$, and hence $A(D, B, C, X) \overline{\wedge} A'(D', B', C', X)$. But these two pencils have a self-corresponding ray DD', and, therefore, the points $(AB, A'B')$, $(AC, A'C')$, $(AX, A'X)$ lie in a straight line. But the third point is precisely $(BC, B'C')$. The converse may be proved in a similar manner. It is very likely that this proof may be found in texts.

Also solved by the PROPOSER.

3208[1926; 385]. Proposed by V. M. Spunar, Chicago, Illinois.

Disprove the following: If a, n are any integers and $a^x \equiv 1 \pmod{n}$ for $x = n-1$ but not when x is an aliquot part of $n-1$, the integer n is a prime. (Lucas, *Theory of Numbers*, I, p. 441.)

SOLUTION BY D. H. LEHMER, University of California.

My object in what follows is not to "solve the problem" but to defend Lucas' theorem by giving a somewhat more evident proof of it.

Suppose that n were composite. Then, if we represent the totient of n by $\phi(n)$ we would have $\phi(n) < n-1$. Now $a^x \equiv 1$ for $x = n-1$ by hypothesis and for $x = \phi(n)$ by Euler's generalization of Fermat's theorem, (since a is evidently prime to n). Let δ be the greatest common divisor of $n-1$ and $\phi(n)$; that is, let δ be such that $n-1 = k\delta$ and $\phi(n) = l\delta$, with k and l integers prime to each other (or with $l = 1$). It is always possible then, to find integers κ and λ such that

$$\kappa k - \lambda l = 1 \text{ and } \delta\kappa k - \delta\lambda l = \delta \text{ or } \kappa(n-1) - \lambda\phi(n) = \delta.$$

Then since $a^{n-1} \equiv a^{\phi(n)} \equiv 1 \pmod{n}$ we have $a^{\kappa(n-1)-\lambda\phi(n)} \equiv a^{\delta} \equiv 1 \pmod{n}$.

But, since δ is clearly an aliquot part of $n-1$, this last relation is contrary to the hypothesis. Hence n is not composite.

Also solved by J. L. REILLY.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor E. T. BELL of the California Institute of Technology has been elected a member of the National Academy of Sciences.

The University of Lemberg has conferred an honorary doctorate on Professor R. A. MULLIKAN of the California Institute of Technology.

Under the title *Mole Philosophy and Other Essays*, E. P. Dutton and Co. have recently published a volume of short essays from the pen of Professor CASSIUS J. KEYSER of Columbia University, mathematician and philosopher. Those acquainted with the author and his earlier publications need hardly be told that the essays exhibit a rare combination of mature wisdom and mellow humor. It is the sort of book one picks up when he has an odd ten minutes of spare time, and puts down reluctantly when the ten minutes have become an hour.

The list of doctorates conferred by American Universities during 1926, as published in the May issue of this Monthly, should have included the

following: GAYLORD M. MERRIMAN, Cincinnati, June, *Sufficient conditions for the summability of double series, with applications to the double Fourier series.*

Professor A. A. Bennett of Lehigh University has been appointed professor of mathematics at Brown University.

Professor A. B. COBLE of the University of Illinois has been appointed professor of mathematics at Johns Hopkins University.

Assistant Professor M. H. INGRAHAM of Brown University has been appointed professor of mathematics at the University of Wisconsin.

Professor C. J. KEYSER of Columbia University has retired, as Adrain professor emeritus.

Dr. LEE H. MCFARLAN, National Research fellow in mathematics stationed at the University of Chicago, has been appointed to an assistant professorship at the University of Washington, Seattle.

Professor F. R. MOULTON of the department of mathematical astronomy of the University of Chicago has recently resigned to go into business.

Dr. C. A. NELSON of Johns Hopkins University has been appointed to an associate professorship of mathematics at Rutgers University, New Brunswick, N. J. At the same institution, Mr. HOWARD PIXLEY, graduate student at the University of Chicago, has been appointed instructor in mathematics.

Assistant Professor J. F. RITT has been promoted to an associate professorship of mathematics at Columbia University.

Assistant Professor J. D. TARMARKIN of Dartmouth College has been appointed assistant professor of Mathematics at Brown University.

Professor R. F. BORDEN of George Washington University died on March 15, 1927.

AN INFORMATION BUREAU FOR APPOINTMENTS

The Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

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MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Summer Meeting of the Association, Madison, Wisconsin, September 5-6, 1927.

Twelfth Annual Meeting, Nashville, Tenn., December, 1927.

The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 13-14.

INDIANA, De Pauw University, April 29-30.

IOWA, University of Iowa, May 6-7.

KANSAS, Topeka, Kan., February 5.

KENTUCKY, Lexington, May 7.

LOUISIANA-MISSISSIPPI, Shreveport, La., March 4-5.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7.

MICHIGAN, April.

MINNESOTA, St. Peter, Minn., May 21.

MISSOURI, St. Louis, Mo., November 25-26.

NEBRASKA, Lincoln, May 14.

OHIO, Columbus, Ohio, April 8.

PHILADELPHIA, Philadelphia, Pa., November 26.

ROCKY MOUNTAIN, Colorado College, April 22-23.

SOUTHEASTERN, Columbia, S. C., April 15-16.

SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12 and November 5.

TEXAS, Not yet determined.

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